A New Splitting for Gauss-Seidel Iteration in Solving One-Dimensional Porous Medium Equations

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ABSTRACT This paper proposes a new matrix splitting technique to improve the Gauss-Seidel iteration to solve one-dimensional porous medium equations. The new splitting uniquely decomposes the coefficient matrix generated from a linear system by further splitting the diagonal part of the matrix into two parts. An unconditionally stable discretization, the implicit finite difference method, is used in the approximation equation formulation. The formulated nonlinear approximation equation is then linearized using the Newton method. The generated system of linear equations is large-sized and sparse, and the coefficient matrix has tridiagonal nonzero entries. A Gauss-Seidel iteration is derived to obtain the solution of the system of linear equations. As a contribution of the paper, the Gauss-Seidel iterative formula is extended to a new method of iteration, known as the Split Gauss-Seidel iteration. The performance comparison between the Gauss-Seidel method and the proposed Split Gauss-Seidel method is conducted by solving several examples of one-dimensional porous medium equations. The experiments compare the number of iterations, the computer time to complete the iteration process and the maximum absolute errors by the two methods. The numerical result shows that the Split Gauss-Seidel method needs fewer iterations and shorter computer time than the Gauss-Seidel method in computing the solutions of the selected examples. Besides that, the accuracy of the solutions has improved when the new matrix splitting technique is implemented. The paper shows that introducing a new matrix splitting technique to the Gauss-Seidel iteration can be a promising numerical matrix method to solve nonlinear parabolic partial differential equations. The theoretical convergence of the method will be studied in the future.

KEYWORDS: porous medium equation; finite difference method; Newton method; Gauss-Seidel; matrix splitting
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INTRODUCTION

The porous medium equation (PME) is one of the important parabolic partial differential equations in many nonlinear physical phenomena such as heat transfer, fluid flow and the diffusion of an ideal gas. The physical phenomena related to the porous medium problems have contributed to applied physics and mathematics research since the early 20th century. Among the earliest discoveries of the PME is the infiltration process of groundwater by Boussinesq in 1903 and the model of isentropic gas flow through a porous medium by Leibenzon and Muscat around 1930 (Vazquez, 2007).

An accurate and efficient PME solution is needed to understand the phenomena better and expand the theory of nonlinear diffusion. The main issue of the equation is the presence of nonlinearity in the partial differential equation, which can cause difficulty in computing the solution. Few numerical methods have been proposed to obtain the approximate solution to the PME, particularly in one-dimension, such as the finite element method (Rabari *et al.*, 2014), moving mesh finite element method (Ngo & Huang, 2017), finite difference method (FDM) (Prajapati *et al.*, 2017) and non-standard FDM (Fazio *et al.*, 2018). However, the efficiency of the computation at the different sizes of a matrix has yet been well studied.

Driven by the literature of the numerical solver of the one-dimensional PME, the paper proposes a new matrix splitting technique that intends to improve the Gauss-Seidel (GS) iteration in solving one-

dimensional PME. The idea and concept of matrix splitting originated from the introduction of the regular splitting theory (Varga, 1962). Since then, many researchers have developed different matrix splitting type iterative methods for solving the system of linear equations. For instance, Jacobi, GS, successive overrelaxation (Saad, 2003), and Hermitian and skew-Hermitian splitting (Bai, 2005). A different linear system requires a different type of matrix splitting iterative method for its efficient solution. Therefore, this paper investigates a suitable matrix splitting iterative method to solve the one-dimensional PME.

The paper uses an unconditionally stable implicit FDMon the general equation of porous medium on a fixed rectangular domain to formulate the approximation equation. From the discretization, the approximation equation obtained is nonlinear. Thus, the paper applies the Newton method to linearize the approximation equation to generate a system of linear equations. Since the system of equations generated from the discretization of finite difference is always large-sized and sparse, the iterative method is the best option for an efficient solution. From the system of linear equations, the GS iteration is derived. As the contribution to the field of the numerical method, the GS iterative formula is extended by introducing a new matrix splitting technique to form a new iteration method known as the Split Gauss-Seidel (SGS) method.

The following sections are outlined as follows: Section 2 discusses the methodology of the original GS method and extends the method to the proposed SGS method. Section 3 presents the numerical experiment of the GS and SGS methods to solve several one-dimensional PME problems. Section 4 concludes the paper with future research.

METHODOLOGY

To begin the formulation of the new SGS iteration method, let consider the general form of a one-dimensional PME,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right), \tag{1}$$

where both α and n are rational numbers (Polyanin & Zaitsev, 2004). The considered approximate solution of Equation (1) is by focusing a rectangular-shaped solution domain with a uniform division of M subinterval in space $0 \le x \le 1$with an equal distance of $h = 1/M$. For the time interval, the paper uses$k = 1/T$ with T is the total time step.

Before discretizing Equation (1) with the unconditionally stable implicit FDM, Equation1 can be implicitly differentiated using the calculus technique into

$$\frac{\partial u}{\partial t} = \alpha\left[u^n \frac{\partial^2 u}{\partial x^2} + nu^{n-1}\left(\frac{\partial u}{\partial x}\right)^2\right], \tag{2}$$

and using the implicit finite difference operators to replace the derivative terms in Equation (2), the corresponding implicit approximation to Equation (1) can be formulated into

$$\begin{aligned} U_{i,j+1} &= \beta\left(U_{i,j+1}^n U_{i+1,j+1} - 2U_{i,j+1}^{n+1} + U_{i,j+1}^n U_{i-1,j+1}\right) \\ &+0.25\beta\left(nU_{i,j+1}^{n-1}U_{i+1,j+1}^2 - 2nU_{i,j+1}^{n-1}U_{i+1,j+1}U_{i-1,j+1} + nU_{i,j+1}^{n-1}U_{i-1,j+1}^2\right) + U_{i,j}\,, \end{aligned} \tag{3}$$

where $\beta = \frac{\alpha\Delta t}{\Delta x^2}$. Since Equation (3) is a nonlinear type of approximation equation that involves three grid points of $U_{i-1,j+1}, U_{i,j+1}$ and$U_{i+1,j+1}$, the linearization of the Newton method is applied to generate the corresponding system of linear equations. Equation (3) is then rewritten into a nonlinear function,

$$\begin{aligned} f_{i,j+1} &= U_{i,j+1} - \beta\left(U_{i,j+1}^n U_{i+1,j+1} - 2U_{i,j+1}^{n+1} + U_{i,j+1}^n U_{i-1,j+1}\right) \\ &-0.25\beta\left(nU_{i,j+1}^{n-1}U_{i+1,j+1}^2 - 2nU_{i,j+1}^{n-1}U_{i+1,j+1}U_{i-1,j+1} + nU_{i,j+1}^{n-1}U_{i-1,j+1}^2\right) - U_{i,j}\,. \end{aligned} \tag{4}$$

When $M-1$ number of interior grid points is considered, the system of nonlinear equations that need to solve has the form of

$$F_{j+1} = 0\,. \tag{5}$$

Now, by applying the Newton method on the system of nonlinear equations (Equation (5)), the transformed system of the linear equation becomes

$$J_{j+1}X_{j+1} = -F_{j+1}\,, \tag{6}$$

where J_{j+1} is the Jacobian matrix concerning $\widehat{U}_{j+1} = \left(U_{1,j+1}, U_{2,j+1}, \ldots, U_{M-2,j+1}, U_{M-1,j+1}\right)$ andX_{j+1} is the corrector vector $\widehat{U}_{j+1}$ that is

$$X_{j+1} = \widehat{U}_{j+1}^{(l+1)} - \widehat{U}_{j+1}^{(l)}\,, \tag{7}$$

with the iteration index (l).

The system of linear equations (Equation (6)) is large-sized and sparse. In addition, the Jacobian matrix has the tridiagonal nonzero entries due to the three grid points to be computed at each time level. Because of the large-sized and sparse system of equations, an iterative method is used as a better option to solve it efficiently. Furthermore, the paper derives the GS iteration among the available iterative methods (Saad, 2003) to solve Equation (6). The derivation of the GS iteration can be made by splitting the coefficient matrix J_{j+1} into

$$J_{j+1} = D_{j+1} + L_{j+1} + V_{j+1}\,, \tag{8}$$

where D is the nonzero entries of a diagonal matrix, L is the lower triangular matrix, and V is the upper triangular matrix, all coefficients need to solve at the time level $j+1$.Using the matrix splitting shown by Equation (8), the iterative formula of the GS method can be derived as (Saad, 2003):

$$X_{j+1}^{(l+1)} = \left(D_{j+1} + L_{j+1}\right)^{-1}\left(-V_{j+1}X_{j+1}^{(l)} - F_{j+1}\right)\,. \tag{9}$$

For the new iterative method, which is the SGS, the paper introduces a new split on Equation (8) by further splitting the diagonal part of the matrix into D_{j+1}^{*} and μ to form

$$J_{j+1} = \left(D_{j+1}^{*} - \mu + L_{j+1}\right) + \left(V_{j+1} + \mu\right)\,, \tag{10}$$

and the iterative formula of the new SGS method becomes

$$X_{j+1}^{(l+1)} = \left(D_{j+1}^{*} - \mu + L_{j+1}\right)^{-1}\left(-\left(V_{j+1} + \mu\right)X_{j+1}^{(l)} - F_{j+1}\right)\,. \tag{11}$$

For the practice of using Equation (11) to solve PME problems, the "Split" coefficient matrix μ is determined by running the developed C program few times until the least number of iterations is obtained. Since the lower and upper triangular matrices in Equation (11) are similar to Equation (9), the convergence of the approximate solutions is guaranteed.

NUMERICAL EXPERIMENTS

For the numerical experiments, the performance comparison between the GS method and the proposed SGS method is conducted by measuring the number of iterations (l_{max}), the computer time to complete the iteration process (seconds) and the maximum absolute errors. To comparel_{max}, the computer time and the maximum errors by the two methods, three initial boundary value problems of the one-dimensional PME are selected as the test problems. Table 1 shows the three problems to be numerically solved by the two methods, which the details can be referred to Polyanin & Zaitsev (2004). In this experiment, the value of α is arbitrarily chosen. The actual value of α can be obtained through the estimation of various physics variables.

Table 1. Selected initial boundary value problems.

Problem	α	n	Exact Solution
1	1	1	$u(x,t) = x + t$
2	1	2	$u(x,t) = \frac{x+1}{2\sqrt{4-t}}$
3	0.5	-2	$u(x,t) = \frac{1}{\sqrt{0.7x - 0.1225t + 1.35}}$

The numerical experiments of the GS and SGS methods to solve the three problems are conducted using five different sizes of large matrices, 256, 512, 1024, 2048, and 4096. These matrices are used to numerically test the convergence of the two iterations with the size of the matrix. Numerical experiments were conducted using a personal computer Intel (R) Core™ i7-3770 CPU 3.40 GHz with 8.00GB of RAM. All the numerical results are recorded and tabulated in Table 2, 3 and 4.

Table 2. Number of iterations required to solve the problems

Problem	Method	Matrix				
		256	512	1024	2048	4096
1	**GS**	48395	169693	587031	1993096	6612931
	SGS	2494	4843	17923	62394	215851
2	**GS**	17308	61658	218147	763998	2630914
	SGS	1765	3445	7213	21695	77848
3	**GS**	24325	81729	265698	882282	2853985
	SGS	1745	3430	9378	30859	100298

Table 3. Computer time required to solve the problems

Problem	Method	Matrix				
		256	512	1024	2048	4096
1	**GS**	19.31	133.84	919.22	6208.25	40998.73
	SGS	1.79	5.18	30.05	207.21	1427.62
2	**GS**	10.88	76.91	557.97	3839.51	26497.28
	SGS	2.24	4.54	24.04	115.54	825.09
3	**GS**	16.70	113.26	767.23	5164.64	33726.52
	SGS	3.39	5.89	27.02	179.00	1211.55

Table 4. Maximum absolute errors produced after the final solutions

Problem	Method	Matrix				
		256	512	1024	2048	4096
1	**GS**	5.33E-07	2.10E-06	7.62E-06	2.67E-05	9.66E-05
	SGS	9.71E-10	1.02E-09	7.67E-08	6.09E-07	2.65E-06
2	**GS**	8.39E-05	8.40E-05	8.43E-05	8.55E-05	8.99E-05
	SGS	8.39E-05	8.39E-05	8.39E-05	8.39E-05	8.40E-05
3	**GS**	2.71E-06	1.86E-06	3.33E-06	1.66E-05	6.10E-05
	SGS	2.97E-06	2.98E-06	2.95E-06	2.61E-06	1.51E-06

Based on the results in Table 2 and 3, the SGS method required fewer iterations and shorter computer time to obtain the solutions of PME problems than the GS method. Then, by computing the percentages of reduction in the number of iterations and computer time by the two methods to obtain the solutions, the study found that the SGS method successfully reduced the number of iterations by 95.72% compared to the GS method. Also, the computer time required to complete the SGS program

has been shortened by 93.58%. Besides that, this significant finding is aided by the level of accuracy of the SGS method. By referring to Table 4, the solutions of Problem 1 and 3 obtained by using the SGS method are more accurate than the GS method. However, for Problem 2, the accuracies by the two methods are similar due to the second-order accuracy of the implicit FDM.

CONCLUSION

In conclusion, the paper successfully presented GS and SGS methods based on the implicit FDM and the Newton method for solving the one-dimensional PME. The performance comparison shows that the SGS method needs fewer iterations and shorter computer time than the GS method. The accuracy of the GS method is also improved with a new matrix splitting technique which shows that the new SGS iteration method can be a promising solver for the nonlinear parabolic partial differential equations. Future research will study the theoretical convergence of the SGS method.

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