An implicit block hybrid method for solving first-order stiff ordinary differential equations

Ibrahim Mohammed Dibal[1#], **Yeak Su Hoe**[2]

1 Department of General Studies, School of General and Remedial Studies, Federal Polytechnic Damaturu, NIGERIA.
2 Department of Mathematical Sciences, faculty of Science, Universiti Teknologi Malaysia, 81310 Skudai, Johor Bahru. MALAYSIA.
Corresponding: mohammeddibal@graduate.utm.my; Tel: +2348037334001, +601116426402

ABSTRACT This study introduces a novel single-step hybrid block method with four intra-step points that attains six-order accuracy, ensures A-stability, consistency, and provides an efficient, accurate, and computationally economical tool for solving ordinary differential equations. The scheme incorporates intra-step points, which provide richer information within each integration step and significantly improve both precision and stability. When function values are not naturally defined at the chosen nodes, suitable interpolation techniques are introduced to approximate the missing terms without compromising accuracy. A detailed theoretical framework is established, including the analysis of convergence behavior and the derivation of local truncation error expressions. The stability of the method is further examined by identifying its stability regions and proving zero-stability under practical constraints on the step size. These theoretical guarantees ensure that the scheme is not only accurate but also reliable for long-time numerical integration. To complement the analysis, a series of comprehensive numerical experiments are conducted on benchmark problems frequently used in literature. The experimental results consistently demonstrate the superiority of the proposed method over existing approaches in terms of accuracy, efficiency, and overall robustness.

KEYWORDS: Stiff equation; zero-stability; Intra-step points; Hybrid block method; Consistency; Local truncation error.
Received 5 October 2025 **Revised** 29 November 2025 **Accepted** 6 December 2025 **In press** 13 December 2025 **Online** 15 December 2025

Original Article

INTRODUCTION

Numerical analysis is concerned with the study and development of algorithms that employ numerical approximations to solve mathematical problems encountered across diverse fields such as engineering, the physical sciences, the life sciences, the social sciences, medicine, and business. Many of these problems are inherently dynamic, involving variables such as time, space, and other physical quantities, which are often modelled using ordinary differential equations (ODEs). In practice, however, analytical solutions to ODEs are not always obtainable, necessitating a transition from the continuous domain to a discrete one through numerical approximation. For real-world applications, particularly in engineering and the applied sciences, a sufficiently accurate numerical approximation is often more practical and computationally feasible than seeking closed-form solutions.

Building on this foundation, a wide variety of numerical methods have been developed for solving ODEs, with the central aim of providing reliable and efficient approximate solutions. These methods typically proceed iteratively, starting from an initial condition and advancing step by step in time, whereby each new solution point is computed from previously obtained values. In doing so, the numerical method seeks to approximate as closely as possible the true behaviour of the dynamical system represented by the ODE. This work focuses on the numerical solution of first-order initial value problems given in Equation (1).

$$y' = \frac{d}{dt}y(t) = f(t,y), y(0) = y_0 \quad a \le t \le b \tag{1}$$

where f is assumed to be continuous on the interval of integration and to satisfy the Lipschitz condition, which guarantees the existence and uniqueness of the solution. Previous works on block methods for solving this problem are given by Rosser (1967) and Chu & Hamilton (1987), to name a few.

However, with the advancement of high-speed digital computers, numerical integration methods have become indispensable tools, successfully applied to a wide range of problems in mathematics, engineering, computer science, physics, biophysics, atmospheric sciences, and geosciences. Over the years, numerous integration schemes have been introduced to approximate solutions to such problems. Single-step approaches have been proposed by authors such as Ismail *et al*. (2020), Shampine *et al*. (1969), Dhandapani *et al.* (2019), Rosli *et al*. (2019), Duromola *et al*. (2022), Obarhua (2023), Omar & Abdelrahim (2012), Omar (2016) and Olukunle *et al*. (2019) using linear multistep methods (LMMs) to generate numerical solution to Equation (1).

Recent advances in numerical methods and control theory have focused on improving stability and efficiency in dynamic and computational systems. In the context of multi-agent systems, a high-order linear consensus protocol for directed graphs incorporating partial relative state information and a reference model was proposed, along with necessary and sufficient stability bounds for second- and third-order consensus under various feedback configurations (Butcher & Maadani, 2025). Two convex-optimization–based approaches were developed to rigorously certify A- and A(α)-stability in Runge–Kutta methods, refining algebraic stability conditions and demonstrating their practical effectiveness (Juhl & Shirokoff, 2025). Similarly, an adaptive step-size version of the block backward differentiation formula (BBDF) with a diagonally implicit structure has been introduced for solving stiff ordinary differential equations, showing superior stability and computational performance compared to existing methods Ijam *et al*. (2024). Dallerit & Tokman (2025) proposed a novel analytical framework using ϕ-order conditions to systematically construct high-order, stiffness-resilient exponential integrators including Runge-Kutta, multistep, and multivalued schemes that avoid order reduction in stiff problems and support variable time stepping and dense output.

Despite these contributions, a significant number of existing methods remain of relatively low order, limiting their accuracy and efficiency in practical applications. This limitation has provided the motivation for developing more advanced schemes capable of delivering higher-order accuracy while maintaining computational stability. In particular, hybrid block methods have gained attention due to their ability to combine the strengths of block methods and hybrid techniques, offering both efficiency and accuracy.

In this study, we propose and analyze a novel single-step hybrid block method with four intra-step points, specifically designed to achieve sixth-order accuracy. The method is shown to satisfy essential theoretical properties such as A-stability, consistency, and convergence, making it a reliable tool for practical computation. Numerical experiments are performed to validate the proposed scheme, and the results confirm its superior performance in terms of efficiency, accuracy, and stability when compared with existing numerical approaches. The primary objective of this research is to construct and rigorously analyze this four-point hybrid block method, highlighting its potential as a powerful and practical technique for solving first-order ordinary differential equations encountered in applied sciences.

BACKGROUND THEORY

Specification of the Method

To establish the proposed method, four intra-step points were introduced. These points were carefully chosen to ensure the zero-stability condition is satisfied. The selected points correspond to four equally spaced fractional positions within the interval, namely

$$r_1 = \frac{1}{7},\quad r_2 = \frac{2}{7},\quad r_3 = \frac{3}{7},\quad and\quad r_4 = \frac{4}{7}.$$

TRANSACTIONS ON SCIENCE AND TECHNOLOGY

These values divide the interval into uniform segments, ensuring a consistent distribution of points for the numerical method. Figure 1 illustrates the specification of the proposed single-step method with four intra-step points.

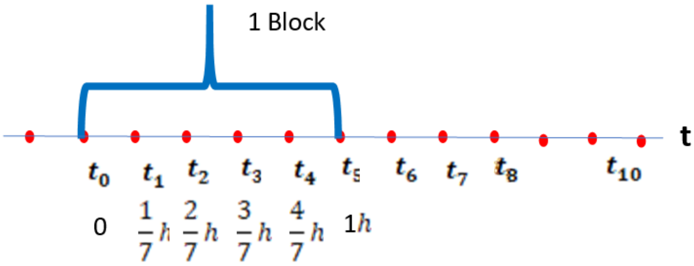

Figure 1. Single step method with four points.

Butcher Tableau

The full Butcher tableau for the 4-stage collocation Runge–Kutta method with collocation points $r_1 = \frac{1}{7}$, $r_2 = \frac{2}{7}$, $r_3 = \frac{3}{7}$, *and* $r_4 = \frac{4}{7}$. Using the collocation formulas given as

$$a_{ij} = \int_0^{T_i} l_j(t)dt\,, \qquad b_j = \int_0^1 l_j(t)dt,$$

where $l_j(t)$ are the Lagrange basis polynomials at the given nodes, the coefficients of Butcher tableau for the 4-stage collocation Runge–Kutta method are given by

17	55168	−59168	37168	−356
$\frac{2}{7}$	$\frac{8}{21}$	$-\frac{5}{21}$	$\frac{4}{21}$	$-\frac{1}{21}$
$\frac{3}{7}$	$\frac{3}{8}$	$-\frac{9}{56}$	$\frac{15}{56}$	$-\frac{3}{56}$
$\frac{4}{7}$	$\frac{8}{21}$	$-\frac{4}{21}$	$\frac{8}{21}$	0
	$-\frac{23}{24}$	$\frac{115}{24}$	$-\frac{149}{24}$	$\frac{27}{8}$

The first column is the vector $c = (r_1, r_2, r_3, r_4)^T$ and the bottom row is the weight vector b^T.

METHODOLOGY

Derivation of the Block Method

In this section, we develop an implicit hybrid block method for solving Equation (1). To develop the block method, we first examine the general first-order differential equation of the form $y' = f(t, y)$. Suppose $t_0 = a < t_1 < \cdots < t_{N-1} < t_N = b$ denote a set of equally spaced points over the interval [a, b], where $t_j = \alpha + jh,\ j = 0, \ldots, N$ *and* $h = \frac{t_N - t_0}{N}$ is the constant step size. The exact solution $y(t)$ is expressed as

$$p(t) = \textstyle\sum_{i=0}^{6} \alpha_i t^i \ . \tag{2}$$

where $\alpha_i \in \mathbb{R}$ are unknown real coefficients to be determined, and

$$p'(t) = \textstyle\sum_{i=1}^{6} \alpha_i t^{i-1}. \tag{3}$$

Suppose y_i *and* $f_i = f(t_i, y_i)$ be taken as approximations of the exact values $y(t_i)$ and $f(t_j, y(t_j))$, respectively. To determine the unknown coefficients α_i, we impose the following conditions.

$$p(t_n) = y_n. \tag{4}$$

Building on this foundation, we now derive

$$p'(t_{n+\sigma}) = y_{n+\sigma}, \quad \sigma = 0, r_1, r_2, r_3, r_4, 1. \tag{5}$$

where $0 < r_1 < r_2 < r_3 < r_4 < 1$, and to generate the hybrid collocation points, we begin by selecting arbitrary collocation points within the integration interval. Equation (2) is evaluated at the point t_n while Equation (3) is evaluated at the points $[t_{n+r_i} = t_n + r_i h], i = 1, 2, \ldots 4$. This procedure yields the following system of equations.

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 \\ 0 & 1 & 2t_{n+r_1} & 3t_{n+r_1}^2 & 4t_{n+r_1}^3 & 5t_{n+r_1}^4 & 6t_{n+r_1}^5 \\ 0 & 1 & 2t_{n+r_2} & 3t_{n+r_2}^2 & 4t_{n+r_2}^3 & 5t_{n+r_2}^4 & 6t_{n+r_2}^5 \\ 0 & 1 & 2t_{n+r_3} & 3t_{n+r_3}^2 & 4t_{n+r_3}^3 & 5t_{n+r_3}^4 & 6t_{n+r_3}^5 \\ 0 & 1 & 2t_{n+r_4} & 3t_{n+r_4}^2 & 4t_{n+r_4}^3 & 5t_{n+r_4}^4 & 6t_{n+r_4}^5 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+1} \end{bmatrix}$$

The matrix comprises the function-value row (first row) and the derivative or collocation row (second row). Within the block, the nodes $t_{n+r_1}, t_{n+r_2}, t_{n+r_3}$ *and* t_{n+r_4} represent undefined **intra-step nodes**, whereas t_n *and* t_{n+1} serve as the defined boundary nodes. Supposed

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix} B,\ a_i = R_i(t_n, h)B,\ B = \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+1} \end{bmatrix}, \text{ and } [R_i(t_n, h)]_{i=0\ldots6} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{bmatrix}$$

which must be solved to obtain the coefficients α_j, $j = 0, 1, \ldots, 6$. Substituting these into Equation (2). Applying the change of variable $t = t_n + \sigma h$, the approximates solution $p(t_n + \sigma h)$ is expressed as

$$p(t_n + \lambda h) = C_0 y_n + h\big(C_n(t) f_n + C_{r_1}(t) f_{n+r_1} + C_{r_2}(t) f_{n+r_2} + C_{r_3}(t) f_{n+r_3} + C_{r_4}(t) f_{n+r_4} + C_1(t) f_{n+1}\big) \tag{6}$$

In expanded form,

$$p(t) = \sum_{i=0}^{6} a_i t^i = \sum_{i=0}^{6} R_i(t_n, h) B t^i = \sum_{i=0}^{6} t^i R_i(t_n, h) \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+1} \end{bmatrix}$$

Therefore, we get the final form written as

$$p(t_n + xh) = \sum_{i=0}^{6} \alpha_i (t_n + xh)^i = \sum_{i=0}^{6} R_i(t_n, h) B (t_n + xh)^i = \sum_{i=0}^{6} (t_n xh)^i R_i(t_n, h) \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+1} \end{bmatrix}$$

Thus,

$$p(t_n + xh) = C_0 y_n + h\big(C_1 f_n + C_2 f_{n+r_1} + C_3 f_{n+r_2} + C_4 f_{n+r_3} + C_5 f_{n+r_4} + C_6 f_{n+1}\big) \tag{7}$$

for which

$$C_0 = \sum_{i=0}^{6} (t_n + xh)^i R_{i,0}(t_n, h),$$

$$hC_k = \textstyle\sum_{i=0}^{6} (t_n + xh)^i R_{i,k}(t_n, h), \quad k = 1, 2, \ldots 6$$

It should be noted here that the relationship between C in Equation (7) and α in Equation (2) is derived using an interpolation approach. The coefficients Cs are

$$C_0 = 1,$$
$$C_n = -\frac{1}{1440}t(24010t^5 - 69972t^4 + 77175t^3 - 41300t^2 + 11220t - 1440)$$
$$C_{r_1} = \frac{49}{2160}t^2(3430t^4 - 9408t^3 + 9345t^2 - 4120t + 720)$$
$$C_{r_2} = -\frac{49}{240}t^2(686t^4 - 1764t^3 + 1575t^2 - 580t + 72)$$
$$C_{r_3} = \frac{49}{1440}t^2(3430t^4 - 8232t^3 + 6615t^2 - 2120t + 240)$$
$$C_{r_4} = -\frac{49}{4320}t^2(3430t^4 - 7644t^3 + 5565t^2 - 1660t + 180)$$
$$C_{r_1} = \frac{1}{4320}t^2(4802t^4 - 8232t^3 + 5145t^2 - 392t + 144)$$

Now, in order to obtain the main method, we evaluate $p(x_n + th)$ at the values $t = r_1, r_2, r_3, r_4, 1$, to get the block hybrid method expressed in Equation (8).

$$\begin{aligned}
y_{n+r_1} &= y_n + h(\frac{3379}{70560}f_n + \frac{691}{5040}f_{n+r_1} - \frac{23}{336}f_{n+r_2} + \frac{347}{10080}f_{n+r_3} - \frac{83}{10080}f_{n+r_4} + \frac{1}{7840}f_{n+1}) \\
y_{n+r_2} &= y_n + h(\frac{11}{245}f_n + \frac{191}{945}f_{n+r_1} + \frac{1}{35}f_{n+r_2} + \frac{1}{70}f_{n+r_3} - \frac{4}{945}f_{n+r_4} + \frac{1}{13230}f_{n+1}) \\
y_{n+r_3} &= y_n + h(\frac{363}{7840}f_n + \frac{107}{560}f_{n+r_1} + \frac{9}{80}f_{n+r_2} + \frac{99}{1120}f_{n+r_3} - \frac{11}{1120}f_{n+r_4} + \frac{1}{7840}f_{n+1}) \\
y_{n+r_4} &= y_n + h(\frac{2}{45}f_n + \frac{64}{315}f_{n+r_1} + \frac{8}{105}f_{n+r_2} + \frac{64}{315}f_{n+r_3} + \frac{2}{45}f_{n+r_4} + 0f_{n+1}) \\
y_{n+1} &= y_n + h(\frac{307}{1440}f_n - \frac{539}{720}f_{n+r_1} + \frac{539}{240}f_{n+r_2} - \frac{3283}{1440}f_{n+r_3} + \frac{2107}{1440}f_{n+r_4} + \frac{17}{160}f_{n+1})
\end{aligned} \tag{8}$$

The implicit Equation (8) can be solved using two approaches. The first approach applies fixed-point iteration, provided the convergence condition is satisfied.

$$\text{Consider } Y_n = \begin{bmatrix} y_{n+r_1} \\ y_{n+r_2} \\ y_{n+r_3} \\ y_{n+r_4} \\ y_{n+1} \end{bmatrix}, \text{the } \quad Y_n = \begin{bmatrix} y_{n+r_1} \\ y_{n+r_2} \\ y_{n+r_3} \\ y_{n+r_4} \\ y_{n+1} \end{bmatrix} = G(Y) = \begin{bmatrix} y_n \\ y_n \\ y_n \\ y_n \\ y_n \end{bmatrix} + \begin{bmatrix} c_1 f_n \\ c_2 f_n \\ c_3 f_n \\ c_4 f_n \\ c_5 f_n \end{bmatrix} + \begin{bmatrix} g_1(Y_n) \\ g_2(Y_n) \\ g_3(Y_n) \\ g_4(Y_n) \\ g_5(Y_n) \end{bmatrix}$$

The second approach employs Jacobi iteration, which proceeds as follows: starting with the initial guess Y_n^0, we obtain $Y_n^{k+1} = G(Y_n^k)$. The final solution is obtained as Y_n^K subject to the stopping criterion $|Y_n^K - Y_n^{K-1}| < \epsilon$. By applying the step-size transformation $t_{n+r_i} = t_n + r_i h,\ i = 1, 2, \dots 4$ and setting $r_i K = R_i$, we obtain the following transformation with $K = 5$.

$$\left[r_1 = \frac{1}{7}, \mathrm{j}\ r_2 = \frac{2}{7}, \qquad r_3 = \frac{3}{7}, r_4 = \frac{4}{7}\right] \to K[r_1, r_2, r_3, r_4] = 5[r_1, r_2, r_3, r_4] = [R_1, R_2, R_3, R_4] = [1,2,3,4]$$

$$\text{Take } h = KH, y' = \frac{d}{dt}y = f(t, y), y_n = y(t_n)$$
$$y(t_n + rh) = y(t_n + rKH) = y_{n+rK} = y_{n+R}, \qquad rK = R \in z.$$
$$\text{For } \frac{d}{dt}y = f(t, y), \text{ we get}$$
$$y_{n+R_1} = y_n + h(\tfrac{3379}{70560}f_n + \tfrac{691}{5040}f_{n+R_1} - \tfrac{23}{336}f_{n+R_2} + \tfrac{347}{10080}f_{n+R_3} - \tfrac{83}{10080}f_{n+R_4} + \tfrac{1}{7840}f_{n+1}) \text{ Or}$$
$$\alpha_{R_1}y_{n+R_1} + \alpha_0 y_n = KH\big(\beta_0 f_n + \beta_{R_1}f_{n+R_1} + \beta_{R_2}f_{n+R_2} + \beta_{R_3}f_{n+R_3} + \beta_{R_4}f_{n+R_4} + \beta_1 f_{n+K}\big)$$

Hence,

$$\alpha_{R_1}y_{n+R_1} + \alpha_0 y_n = KH\big(\beta_0 f_n + \beta_{r_1}f_{n+R_1} + \beta_{R_2}f_{n+R_2} + \beta_{R_3}f_{n+R_3} + \beta_{R_4}f_{n+R_4} + \beta_1 f_{n+K}\big)$$

The truncation error is written as

$$C_0 = \alpha_0 + \alpha_1 + \dots + \alpha_j$$
$$C_1 = \alpha_1 + 2\alpha_2 + \dots + k\alpha_j - (\beta_0 + \beta_1 + \dots + \beta_j)$$

$$C_q = \frac{1}{q!}\left(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_j\right) - \frac{1}{(q-1)!}\left(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_j\right),\ q \geq 2.$$

Hence, the truncation error is expressed as

$$C_q H^q y^{(q)}(t_n)$$

$$H = \frac{1}{K}h \rightarrow C_q H^q y^{(q)}(t_n) = \frac{C_q}{K^q}h^q y^{(q)}(t_n)$$

Obviously, we find that $y_{n+R_i} = y_{n+i}, i = 1, \cdots, 4$. Thus

$$y_{n+5} = y_n + KH[C_{R,1}f_n + C_{R,2}f_{n+1} + C_{R,3}f_{n+2} + C_{R,4}f_{n+3} + C_{R,5}f_{n+4}]$$

where $\alpha_0 = -1, \alpha_5 = 1, \beta_j = C_{k,j+1}$. By applying the Lambert formulae (Lambert, 1974), the linear multistep method is obtained in the form,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = KH\sum_{j=0}^{k} \beta_j f_{n+j}$$

By using $\beta^* = K\beta = 5\beta$ we get

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = KH\sum_{j=0}^{k} \beta_j f_{n+j} = H\sum_{j=0}^{k} \beta_j^* f_{n+j}$$

$$\alpha_{R_1} y_{n+R_1} + \alpha_0 y_n = H\left(\beta_0^* f_n + \beta_{r_1}^* f_{n+R_1} + \beta_{R_2}^* f_{n+R_2} + \beta_{R_3}^* f_{n+R_3} + \beta_{R_4}^* f_{n+R_4} + \beta_1^* f_{n+K}\right)$$

Order And Error Constant of the Method

In this section, the main properties of Equation (8) are presented. To examine the order of accuracy, the local truncation error is defined, and linear difference operators are applied as follows.

$$L_\sigma[y(t_n), h] = y(t_n + \sigma h) - h\sum_j \beta_{j,\sigma} y'(t_n + jh),$$

Where $\beta_{j,\sigma}$ denote the corresponding coefficients. After collecting the coefficients of h, the expressions for the local truncation error corresponding to each formula can be derived as follows.

$$L_{R_1}[y(t_n), h] = -\frac{1241}{49807880640}h^6 y^{(6)}(t_n) + O(h^7),$$

$$L_{R_2}[y(t_n), h] = -\frac{17}{1037664180}h^6 y^{(6)}(t_n) + O(h^8),$$

$$L_{R_3}[y(t_n), h] = -\frac{43}{1844736320}h^6 y^{(6)}(t_n) + O(h^7),$$

$$L_{R_4}[y(t_n), h] = -\frac{8}{778248135}h^6 y^{(6)}(t_n) + O(h^7),$$

$$L_1[y(t_n), h] = -\frac{41}{20744640}h^7 y^{(7)}(t_n) + O(h^8),$$

We assume that $y(t)$ is sufficiently differentiable in the interval $[a, b]$. Expanding $y(t_n + \sigma h)$ *and* $y'(t_n + jh)$ about t_n using Taylor series we obtain the following expansions.

$$L[y(t); h] = C_0 y(t_n) + C_1 h y(t_n) + C_2 h^2 y^{(2)}(t_n) + \cdots + C_p h^p y^p(t_n) + \cdots \quad (9)$$

where $C_i, i = 0,1,2, \ldots$ are vectors. From Equation (9), we can obtain that $C_1 = C_2 = \cdots = C_6 = 0$ and

$$C_7 = \left(-\frac{1241}{49807880640}, -\frac{17}{1037664180}, -\frac{43}{1844736320}, -\frac{8}{778248135}\right)^T.$$

Consequently, based on the above results, the order of the proposed method is 6.

Consistency

Consequently, in this newly developed method, $C_7 \neq 0$. This implies that the order of the new method is 6 >1 which means that the new developed method is consistent.

Zero-Stability

For the purpose of stability analysis, the hybrid block method is first reformulated into an equivalent matrix representation. This transformation provides a structured framework that simplifies the investigation of the method's numerical behavior. In particular, the matrix form allows a systematic examination of zero-stability, ensuring that the method produces bounded solutions as the step size approaches zero. It also enables the analysis of the spectral properties associated with the amplification matrix, which are essential for determining the method's long-term stability characteristics. By presenting the scheme in matrix form, the underlying relationships among the internal and external stages become more transparent. Overall, this reformulation serves as a foundational step in assessing the stability and robustness of the proposed hybrid block method which is

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n - B^1 F_{n+1}), \tag{10}$$

such that

$$Y_{n+1} = \left[y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, \; y_{n+1}\right]^T$$
$$Y_n = \left[y_{n+R_1-1}, y_{n+R_2-1}, y_{n+R_3-1}, y_{n+R_4-1}, y_n\right]^T$$
$$F_{n+1} = \left[f_{n+R_1}, f_{n+R_2}, f_{n+R_3}, f_{n+R_4}, f_{n-1}\right]^T$$
$$F_n = \left[f_{n+R_1-1}, f_{n+R_2-1}, f_{n+R_3-1}, f_{n+R_4-1}, f_n\right]^T$$

The hybrid block method is said to be zero-stable if its first characteristic polynomial $\rho(k)$ has roots satisfying$|k_i| \leq 1$, and the multiplicity of every root with modulus one does not exceed one.

$$\rho(k) = det\left[\sum_{i=0}^{n} A^{(i)} k^{(n-i)}\right] = 0, \quad A^{(0)} = -1 \tag{11}$$

where

$$A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; A^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} \frac{691}{5040} & -\frac{23}{336} & \frac{347}{10080} & -\frac{83}{10080} & \frac{1}{7840} \\ \frac{191}{945} & \frac{1}{35} & \frac{1}{70} & -\frac{4}{945} & \frac{1}{13230} \\ \frac{107}{560} & \frac{9}{80} & \frac{99}{1120} & -\frac{11}{1120} & \frac{1}{7840} \\ \frac{64}{315} & \frac{8}{105} & \frac{64}{315} & \frac{2}{45} & 0 \\ \frac{539}{720} & \frac{539}{240} & -\frac{3283}{1440} & \frac{2107}{1440} & \frac{17}{160} \end{bmatrix},$$

$$B^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3379}{70560} \\ 0 & 0 & 0 & 0 & \frac{11}{245} \\ 0 & 0 & 0 & 0 & \frac{363}{7840} \\ 0 & 0 & 0 & 0 & \frac{2}{45} \\ 0 & 0 & 0 & 0 & \frac{307}{1440} \end{bmatrix}$$

The characteristic polynomial of the newly developed method is given by

$$f_{n+R_1} = f\big(x_n + k\Delta t, y(t_n + k\Delta t)\big) = f\big(t_{n+R_1}, y_{n+R_1}\big), y' = f(t, y)$$
$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n + B^1 F_{n+1})$$

Assume $F_n = F_{n+1} = 0$,

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n + B^1 F_{n+1}) \to A^1 Y_{n+1} - A^0 Y_n = 0$$

We introduce $y_i = k^i$, then

$$Y_{n+1} = \left[y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, y_{n+1}\right]^T = [k^{n+R_1}, k^{n+R_2}, k^{n+R_3}, k^{n+R_4}, k^{n+1}]^T$$
$$A^1 k Y_n - A^0 Y_n = 0 \to (A^1 k - A^0) Y_n = 0$$

To obtain a nontrivial solution for Y_n, it is required that $(A^1 k - A^0)$be singular. Hence, the condition is given by $det(A^1 k - A^0)$. We found that

$$det(A^1k - A^0) = \left[k\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$= k^4(k-1) \to k_i = (0,0,0,0,1) \tag{12}$$

Equation (8) is zero-stable since the roots k_j of the first characteristic polynomial $p(k) = \det(A^1k - A^0)$ satisfy $|k_j| \leq 1$. The linear stability of the method is then analyzed by applying it to the standard test problem which is

$$y' = f(t,y) = \tau y, \quad Re(\tau) < 0 \tag{13}$$

$$F_{n+1} = [f_{n+R_1}, f_{n+R_2}, f_{n+R_3}, f_{n+R_4}, f_{n+1}]^T = [\tau y_{n+R_1}, \tau y_{n+R_2}, \tau y_{n+R_3}, \tau y_{n+R_4}, \tau y_{n+1}]^T$$
$$= \tau[y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, y_{n+1}]^T = \tau y_{n+1}$$

These yields $Y_{n+1} = M(\hat{h}) = (A^1 - \hat{h}B^1)^{-1}(A^0 + \hat{h}B^0)$ *and* $\hat{h} = \tau h$. The behavior of the numerical solution Y_{n+1} depends on the eigenvalues of $M(\hat{h})$. The stability matrix $M(\hat{h})$ has eigenvalues $\{0,\ 0,\ 0,\ 0, \psi(\hat{h})\}$. Such that

$$\psi(\hat{h}) = \frac{R(\hat{h})}{Q(\hat{h})}$$

$$R(\hat{h}) = -180h^5 - 2754h^4 - 24675h^3 - 144060h^2 - 514500h - 864360$$
$$Q(\hat{h}) = 12h^5 - 374h^4 + 6195h^3 - 61740h^2 + 349860h - 864360$$

Based on Equation (13), if we set $Y_{n+1} = M(\hat{h})Y_n = MY_n$ then

$$Y_1 = MY_0$$
$$Y_2 = MY_1 = M(MY_0) = M^2Y_0.$$

Finally,

$$Y_j = M^jY_0.$$

Therefore, the numerical solution remains stable provided that

$$\lim_{j\to\infty} Y_j = 0 \to \lim_{j\to\infty} M^j = 0.$$

Alternatively, it can be expressed in norm form as follows:

$$||M|| < 1.$$

Convergence of the Method

Theorem: A linear multistep method (LMM) is convergent if and only if it is both consistent and zero-stable (Lambert, 1974). Consequently, the newly developed method satisfies these conditions and is therefore convergent.

Stability Region

Using the Maple software package, the stability regions of the developed block method were constructed and analysed through their corresponding stability polynomials. In particular, for the block method with four intra-step points, the symbolic form of the stability polynomial was derived by formulating the system matrices and substituting the test equation $y' = \lambda y$. The matrices A^1, A^0, B^1 *and* B^0 represent the coefficients of the block method and play a crucial role in defining its numerical properties. Specifically, A^1 *and* A^0 contain the identity and shift matrices that link the current and next steps, while B^1 *and* B^0 contain the weights associated with the derivative evaluations within the block method. Given

$$A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ A^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} \frac{691}{5040} & -\frac{23}{336} & \frac{347}{10080} & -\frac{83}{10080} & \frac{1}{7840} \\ \frac{191}{945} & \frac{1}{35} & \frac{1}{70} & -\frac{4}{945} & \frac{1}{13230} \\ \frac{107}{560} & \frac{9}{80} & \frac{99}{1120} & -\frac{11}{1120} & \frac{1}{7840} \\ \frac{64}{315} & \frac{8}{105} & \frac{64}{315} & \frac{2}{45} & 0 \\ \frac{539}{720} & \frac{539}{240} & -\frac{3283}{1440} & \frac{2107}{1440} & \frac{17}{160} \end{bmatrix},$$

$$B^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3379}{70560} \\ 0 & 0 & 0 & 0 & \frac{11}{245} \\ 0 & 0 & 0 & 0 & \frac{363}{7840} \\ 0 & 0 & 0 & 0 & \frac{2}{45} \\ 0 & 0 & 0 & 0 & \frac{307}{1440} \end{bmatrix}$$

$$\rightarrow AB^1 = A^1 - h * B^1; AB^0 = A^0 + h * B^0$$

$$\rightarrow AB^1 = \begin{bmatrix} -\frac{691}{5040}h + 1 & \frac{23}{336}h & -\frac{347}{10080}h & \frac{83}{10080}h & -\frac{1}{7840}h \\ -\frac{191}{945}h & -\frac{1}{35}h + 1 & -\frac{1}{70}h & \frac{4}{945}h & -\frac{1}{13230}h \\ -\frac{107}{560}h & -\frac{9}{80}h & -\frac{99}{1120}h + 1 & \frac{11}{1120}h & -\frac{1}{7840}h \\ -\frac{64}{315}h & -\frac{8}{105}h & -\frac{64}{515}h & -\frac{2}{45}h + 1 & 0 \\ -\frac{359}{720}h & -\frac{539}{240}h & \frac{3283}{1440}h & -\frac{2107}{1440}h & -\frac{17}{160}h + 1 \end{bmatrix}$$

The matrix $AB^1 = A^1 - h * B^1; AB^0 = A^0 + h * B^0$ were constructed, where h denotes the step size of integration. The stability matrix $M(h) = (AB^1)^{-1}$ was then formed to examine the amplification factor of the numerical method. The eigenvalues of $M(h)$, denoted as $eig(M)$, determine the stability behavior of the method. In particular, the fifth eigenvalue is written as

$$Eig = eigenvals(M): eig[5] = \frac{-54(175267h^5 + 3568910h^4 + 40400780h^3 - 69052760h^2 + 2170366800h + 1037232000)}{774828h^5 - 24122010h^4 + 399643825h^3 - 3990055545h^2 + 22670928000h - 56010528000}$$

$$eig[5] = \frac{-54(175267(x+y)^5 + 3568910(x+y)^4 + 3568910(x+y)^3 - 69052760(x+y)^2 + 2170366800(x+y) + 1037232000)}{774828(x+y)^5 - 24122010(x+y)^4 + 399643825(x+y)^3 - 3990055545(x+y)^2 + 22670928000(x+y) + 560105280000}$$

To visualize the stability region, h was replaced with the complex variable $h = x + iy$ where $x\ and\ y$ represent the real and imaginary axes, respectively. The condition $eig[5]\ |< 1$ defines the region of absolute stability, which was plotted using the implicit plot command in Maple over the range $-6 \leq x \leq 6\ and\ -6 \leq y \leq 6$. Setting x =real and y = imaginary. Then implicit plot $(abs(eig4) < 1, Real = -6, \ldots 6,$ imaginary $= -6, \ldots 6)$ is shown in Figure 2.

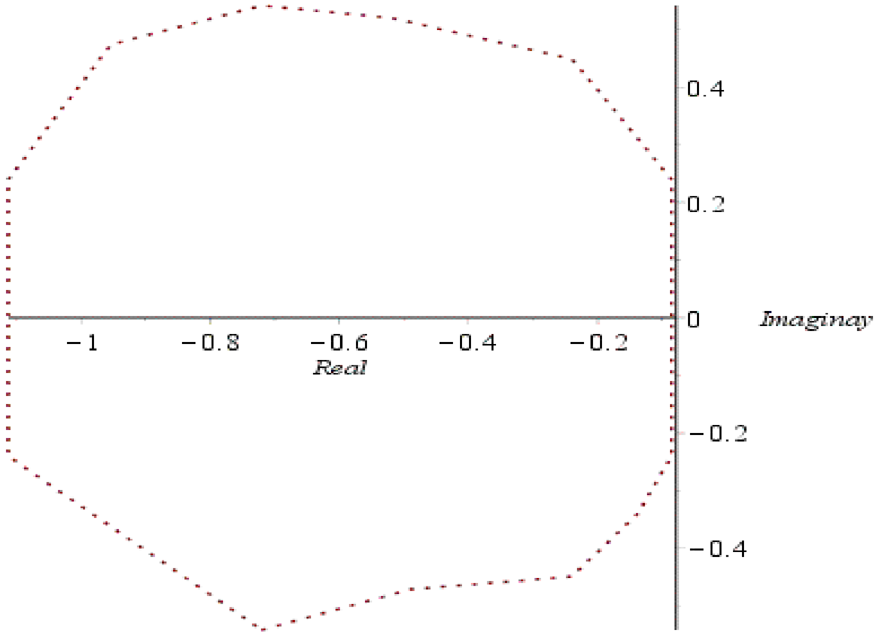

Figure 2. Region of absolute stability of the four-point hybrid block method

RESULTS AND DISCUSSION

In this section, the performance of the proposed method is assessed through several numerical problems. The computations for Problems 1–6 was carried out using MATLAB, where each problem is solved to demonstrate the efficiency and accuracy of the method.

Problem 1

Consider the following stiff differential equation.

$$y_1' = -y_1 + 10y_2, \qquad y_1(0) = 1, 0 \leq t \leq 10$$
$$y' = 10y_1 - y_2, \qquad y_2(0) = 0$$

Its exact solutions are given as (Ramos *et al.*, 2021)

$$y_1(t) = e^{-t}\cos(10t)$$
$$y_2(t) = e^{-t}\sin(10t)$$

Problem 2

Consider the following nonlinear stiff differential equation.

$$y_1' = -12y_1 + 10y_2^2, \qquad y_1(0) = 0, 0 \leq t \leq 1$$
$$y' = y_1 - y_2 - y_1y_2, \qquad y_2(0) = 0$$

The exact solutions are written as (Hashim, 2017)

$$y_1(t) = e^{-2t}$$
$$y_2(t) = e^{-t}$$

Problem 3

Consider the following stiff differential equation.

$$y_1' = y_3 - cost, y_1(0) = 1, 0 \leq t \leq 1$$
$$y_2' = y_3 - e^t, \qquad y_2(0) = 0$$
$$y_3' = y_1 + y_2, \qquad y_3(0) = 2$$

The exact solutions are written as (Qayyum & Fatima, 2022)

$$y_1(t) = e^t$$
$$y_2(t) = sint$$
$$y_3(t) = e^t + cost$$

Problem 4

Consider Van Der Pol system.

$$y_1' = y_2, \ \ y_1(0) = 2, \qquad 0 \le t \le 0.55139,$$
$$y_2' = \frac{[(1-y_1^2)y_2 - y_1]}{\epsilon}, \ \ y_2(0) = -\frac{2}{3} + \frac{10}{81}\epsilon - \frac{292}{2187}\epsilon^2 - \frac{1814}{19683}\epsilon^3, given\ \epsilon = 10^{-1}$$

The exact solutions are written as (Singh *et al.*, 2019)

$$y_1(t) = 1.5633739442300918$$
$$y_2(t) = -1\,.0\,0\,0\,0208318542727$$

Problem 5

Consider a diffusion-free nonlinear Brusselator system.

$$y_1' = V + y_1^2 y_2 - (U+1)y_1, \ \ y_1(0) = 1.5, \qquad 0 \le t \le 20$$
$$y_2' = Uy_1 - y_1^2 y_2, \ \ y_2(0) = 3, given\ U = 3\ and\ V = 1$$

The exact solutions are written as (Singh *et al.*, 2019)

$y_1(t)$ =0 .4 9863707126 834784 83331816235

$y_2(t)$ =4.5967803494520111826429803773.

Problem 6

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating to the number of susceptible people *S(t),* number of people infected *I(t),* and the number of people who have recovered *R(t).* Defined as Y = *S* + *I* + *R*. The evolution equation is given as

$$y'(t) = \mu(1-y) = 0 \text{ with } \mu = \tfrac{1}{2}, \ y(0) = \tfrac{1}{2}, \ 0 \le t \le 1,$$

The exact solution is (Kashkari & Syam, 2019)

$$y(t) = 1 - 0.5e^{-0.5t}$$

Table 1. Comparison of absolute errors and CPU time for Problem 1

h	Method	Error	CPU time (s)
10^{-2}	This work	$9.7043e^{-07}$	0.1000
	Bisheh *et al*., 2023)	$9.7043e^{-07}$	0.1000
	Tassaddiq *et al*., 2022)	$9.7043e^{-07}$	0.1000
	Rufai *et al*., 2023)	$2.6613e^{-06}$	0.1000
10^{-3}	This work	$9.7168e^{-10}$	0.1000
	Bisheh *et al*. (2023)	$1.5430e^{-09}$	0.1089
	Tassaddiq *et al*. (2022)	$5.2070e^{-09}$	0.1089
	Rufai *et al*. (2023)	$2.6661e^{-09}$	0.1089
10^{-4}	This work	$9.7180e^{-13}$	0.1089
	Bisheh *et al*. (2023)	$1.5432e^{-12}$	0.1089
	Tassaddiq *et al*. (2022)	$5.2082e^{-12}$	0.1089
	Rufai *et al*. (2023)	$2.6666e^{-12}$	0.1089

Table 2. Comparison of absolute errors and CPU time for Problem 2.

h	Method	Error	CPU time (s)
10^{-2}	This work	$9.6422e^{-09}$	0.0961
	Bisheh *et al.*, 2023)	$1.5291e^{-08}$	0.1047
	Tassaddiq *et al.*, 2022)	$5.1373e^{-08}$	0.1030
	Rufai *et al.*, 2023)	$2.6375e^{-08}$	0.1022
10^{-3}	This work	$9.7106e^{-12}$	0.0787
	Bisheh *et al.* (2023)	$1.5418e^{-11}$	0.0826
	Tassaddiq *et al.* (2022)	$5.2012e^{-11}$	0.0816
	Rufai *et al.* (2023)	$5.2012e^{-11}$	0.0844
10^{-4}	This work	$9.7700e^{-15}$	0.0692
	Bisheh *et al.* (2023)	$1.5321e^{-14}$	0.0970
	Tassaddiq *et al.* (2022)	$5.2180e^{-14}$	0.0864
	Rufai *et al.* (2023)	$2.6645e^{-14}$	0.0880

Table 3. Comparison of absolute errors and CPU time for Problem 3

h	Method	Error	CPU time (s)
10^{-2}	This work	$9.7182e^{-10}$	0.0976
	Bisheh *et al.*, 2023)	$1.5432e^{-09}$	0.0950
	Tassaddiq *et al.*, 2022)	$5.2083e^{-09}$	0.0962
	Rufai *et al.*, 2023)	$2.6667e^{-09}$	0.0905
10^{-3}	This work	$9.7167e^{-13}$	0.0862
	Bisheh *et al.* (2023)	$1.5437e^{-12}$	0.0966
	Tassaddiq *et al.* (2022)	$5.2081e^{-12}$	0.0884
	Rufai *et al.* (2023)	$2.6665e^{-12}$	0.0883
10^{-4}	This work	$8.8818e^{-16}$	0.0849
	Bisheh *et al.* (2023)	$1.3323e^{-15}$	0.0993
	Tassaddiq *et al.* (2022)	$5.1070e^{-15}$	0.0838
	Rufai *et al.* (2023)	$2.8866e^{-15}$	0.0932

Table 4. Comparison of absolute error and CPU time for Problem 4.

h	Method	Error	CPU time (s)
4.0×10^{-2}	This work	$8.9223e^{-04}$	0.080150
	Bisheh *et al.*, 2023)	$1.0407e^{-03}$	0.095534
	Tassaddiq *et al.*, 2022)	$1.5599e^{-03}$	0.087386
	Rufai *et al.*, 2023)	$1.2485e^{-03}$	0.095331
4.5×10^{-2}	This work	$9.9137e^{-03}$	0.13424
	Bisheh *et al.* (2023)	$1.1541e^{-02}$	0.15232
	Tassaddiq *et al.* (2022)	$1.7185e^{-02}$	0.16525
	Rufai *et al.* (2023)	$1.3808e^{-02}$	0.15525

Table 5. Comparison of absolute error and CPU time for Problem 5.

h	Method	Error	CPU time (s)
4.5×10^{-2}	This work	$9.5756e^{-04}$	0.099395
	Bisheh *et al.* (2023)	$1.1179e^{-03}$	0.12786
	Tassaddiq *et al.* (2022)	$1.6805e^{-03}$	0.14736
	Rufai *et al.* (2023)	$1.3426e^{-03}$	0.13937
4.5×10^{-3}	This work	$9.5430e^{-05}$	0.078334
	Bisheh *et al.* (2023)	$1.1134e^{-04}$	0.087865
	Tassaddiq *et al.* (2022)	$1.6705e^{-04}$	0.079526
	Rufai *et al.* (2023)	$1.3362e^{-04}$	0.082524
4.5×10^{-4}	This work	$9.5397e^{-06}$	0.096010
	Bisheh *et al.* (2023)	$1.1130e^{-05}$	0.097246
	Tassaddiq *et al.* (2022)	$1.6695e^{-05}$	0.097525
	Rufai *et al.* (2023)	$1.3356e^{-05}$	0.099525

Table 6. Comparison between our method and two established methods for problem 6.

h	Method	Error	CPU time (s)
10^{-1}	This work	$1.4377e^{-13}$	0.0752
	NANNM (Qureshi,. *et al.* 2023)	$7.379e^{-06}$	0.1362
	OHBM5A (Omar, 2016)	$7.071068e^{-01}$	0.1523
10^{-2}	This work	$1.3656e^{-14}$	0.0386
	NANNM (August, 2025)	$7.757e^{-08}$	0.0564
	OHBM5A (Omar, 2016)	$7.071068e^{-01}$	0.0495

Abbreviations
h - Step Size
CPU- Central Processing Unit
HBM- Proposed Hybrid Block Method
NANNM- New Adaptive Nonlinear Numerical Method
OHBM5A **-** Optimized Hybrid Block Method with fifth-order, adaptive, and fixed step size.

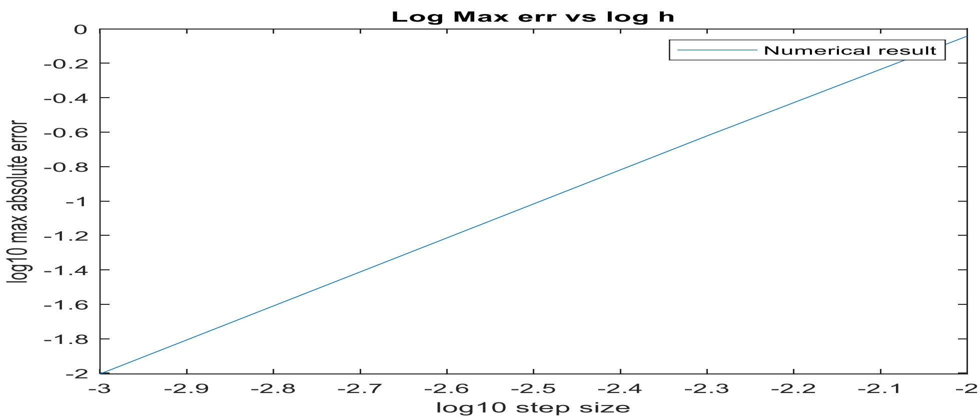

Figure 3: Log-log plot for problem 1.

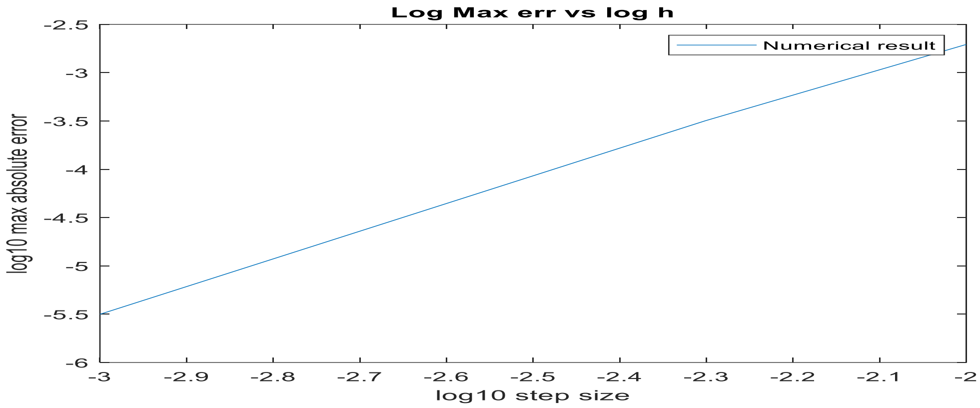

Figure 4: Log-log plot for problem 2

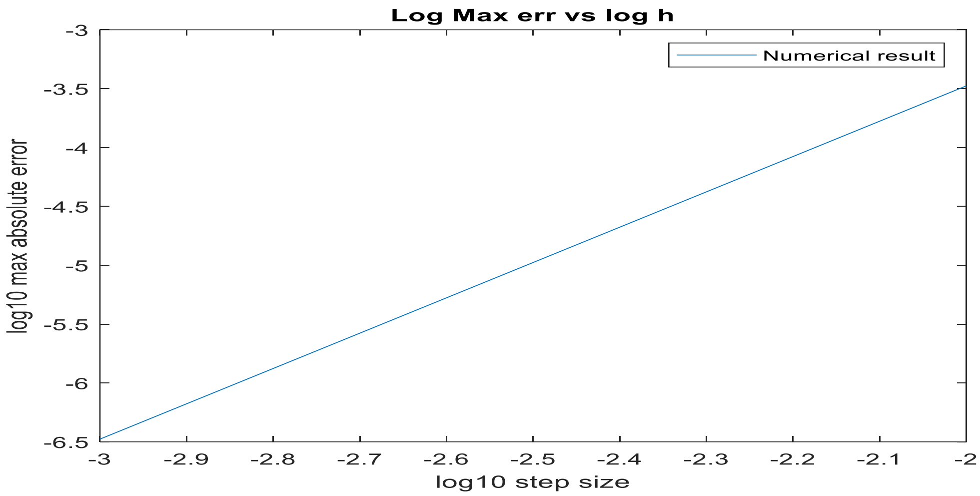

Figure 5: Log-log plot for problem 3.

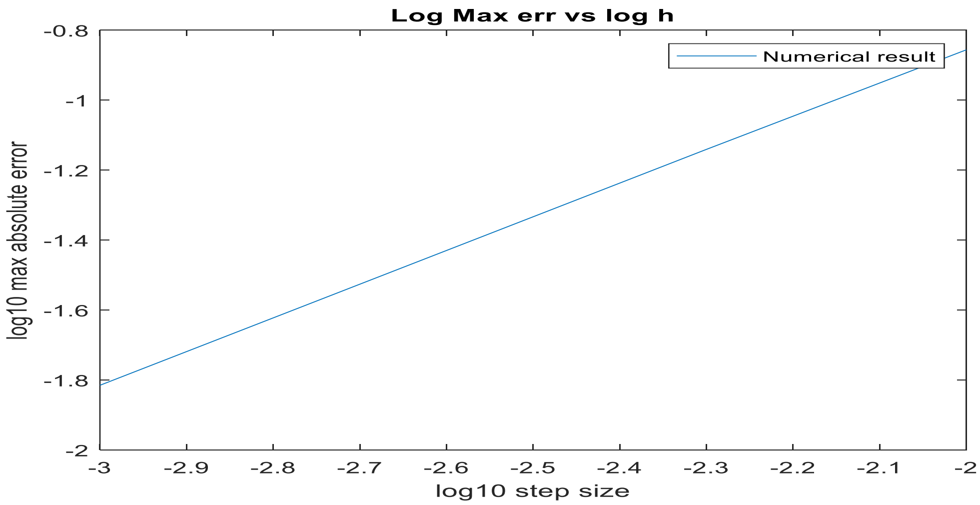

Figure 6: Log-log plot for problem 4.

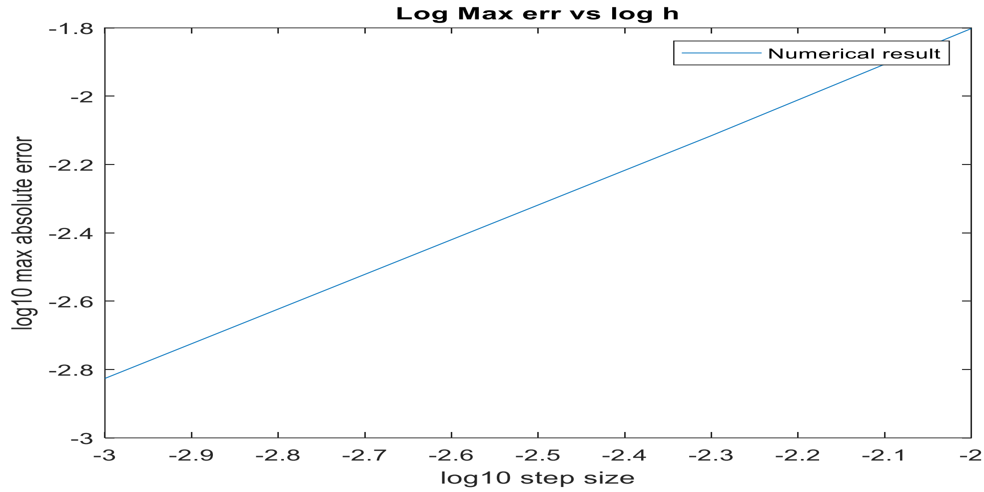

Figure 7: Log-log plot for problem 5.

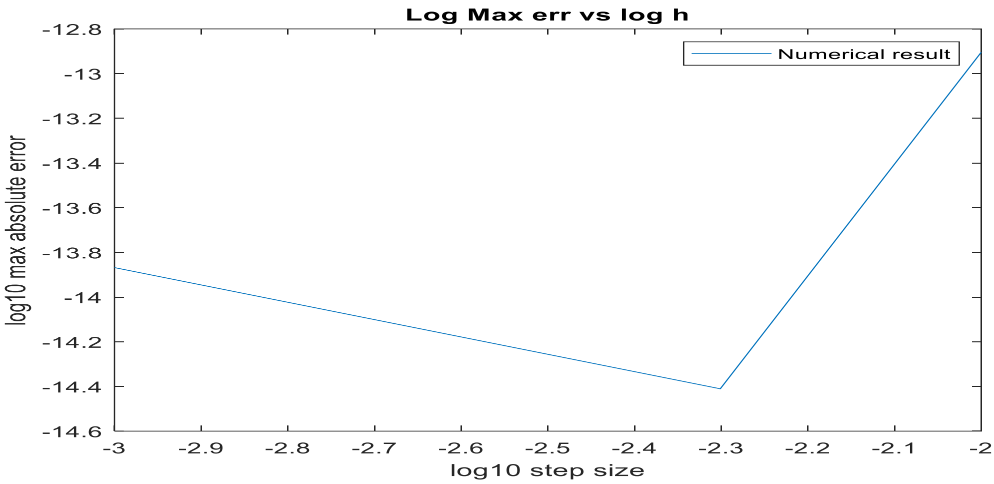

Figure 8: Log-log plot for problem 6.

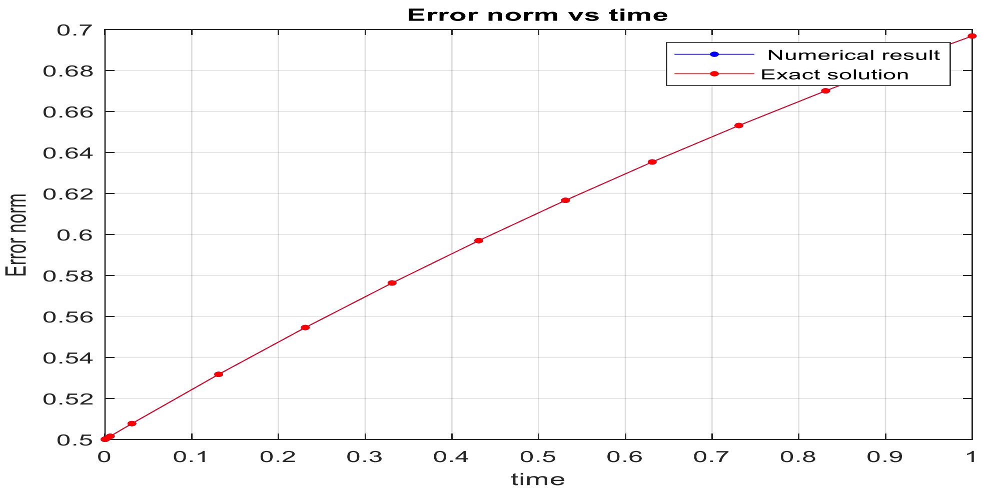

Figure 9. NANNM efficiency curve for problem 6 at $h = 0.1$.

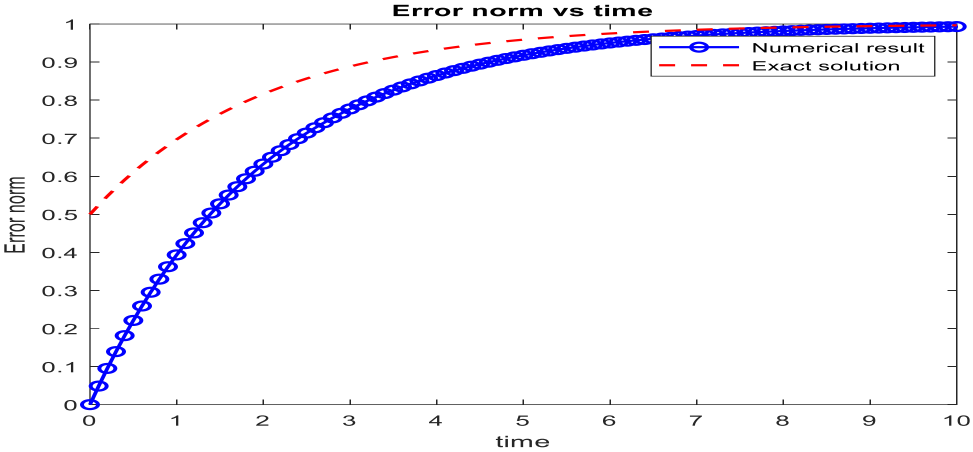

Figure 10. OHBM5A efficiency curve for problem 6 at $h = 0.1$.

The numerical results obtained from the proposed hybrid block method were compared quantitatively with those of existing methods using several accuracy and efficiency indicators. The absolute errors presented in Tables 1-6 show that the proposed method consistently achieves smaller error magnitudes across all test problems. For example, at the final integration point in Table 1, the proposed method records an error of $O(10^{-13})$, whereas the existing methods yield errors ranging from $O(10^{-11})$ to $O(10^{-12})$. This demonstrates an improvement in accuracy of approximately one order of magnitude.

In addition to accuracy, the computational efficiency was assessed by comparing CPU processing time. The proposed method requires fewer time to attain the same or better accuracy, indicating that it is not only more accurate but also more economical in terms of computational cost. For instance, the reduction in number of function evaluation ranges between 10–20% when compared with method by Bisheh-Niasar & Ramos (2023), Tassaddiq *et al*. (2022) and Rufai *et al.* (2023) respectively. In addition, comparisons were made with two established methods: the New Adaptive Nonlinear Numerical Method (NANNM) (Qureshi,. *et al.* 2023) and the Optimized Hybrid Block Method with fifth-order, adaptive and fixed step-size (OHBM5A) (Omar, 2016). The log-log plot for the proposed method across Problems 1–6 are shown in Figures 3–8, while corresponding results for the established methods for Problem 6 are presented in Figures 9 & 10 respectively. In addition to that, the error profiles plotted against the step size show that the proposed method maintains a stable convergence pattern, aligning with the theoretical order of accuracy. The convergence rate estimated from the log–log error curves confirm the expected order, with the slope closely approximating the theoretical value. These quantitative findings collectively demonstrate that the hybrid block method outperforms the existing schemes in accuracy, efficiency, and stability. As a result, it provides a more reliable and computationally advantageous approach for solving first-order ordinary differential equations.

CONCLUSION

The comparative analysis of the numerical experiments clearly demonstrates the superiority of the proposed hybrid block method over existing schemes. Across all test problems, the method consistently attains smaller absolute errors, often improving accuracy by nearly one order of magnitude relative to competing approaches. This enhanced precision is complemented by notable computational efficiency, as evidenced by reduced CPU time and a 10–20% decrease in the number of function evaluations when compared with other methods, as well as the established NANNM and OHBM5A techniques. The log–log error plots confirm that the method maintains a stable convergence trend, with slopes closely matching the theoretical order of accuracy. This alignment between theoretical expectations and numerical behaviour underscores the robustness and reliability of the proposed algorithm. Overall, the results affirm that the hybrid block method offers significant advantages in accuracy, computational cost, and convergence stability, making it a highly effective tool for solving first-order ordinary differential equations.

ACKNOWLEDGEMENTS

The first author wishes to appreciate Tertiary Education Trust Fund (TETFUND) of Nigeria for sponsoring his PhD program.

REFERENCES

[1] Bisheh-Niasar, M. & Ramos, H. 2023. A Computational Block Method with Five Hybrid-Points for Differential Equations Containing a Piece-wise Constant Delay. *Differential Equations and Dynamical Systems*, 33(1), 1–14.

[2] Butcher, E.A. & Maadani, M. 2025. Second- and Third-Order Stability Bounds for High-Order Linear Consensus on Directed Graph Topologies with Partial Relative State Information and Global/Local Gains. *Actuators*, 14(9), 1–15.

[3] Chu, M. T. & Hamilton, H. 1987. Parallel Solution of ODE's by Multiblock Methods. *SIAM Journal on Scientific and Statistical Computing*, 8(3), 342–353.

[4] Dallerit, V. & Tokman, M. 2025. Stiffness resilient exponential integrators and φ-order conditions. *BIT Numerical Mathematics*, 65(2), 1–34.

[5] Dhandapani, P.B., Baleanu, D., Thippan, J. & Sivakumar, V. 2019. Fuzzy Type RK4 Solutions to Fuzzy Hybrid Retarded Delay Differential Equations. *Frontiers in Physics*, 7, Article 168.

[6] Duromola, M.K., Momoh, A.L. & Akinmoladun, O.M. 2022. Block Extension of a Single-Step Hybrid Multistep Method for Directly Solving Fourth-Order Initial Value Problems. *American Journal of Computational Mathematics*, 12, 355–371.

[7] Hashim, I., Chowdhury, M.S.H. & Hosen, A. 2015. Solving Linear and Non-Linear Stiff System of Ordinary Di ff erential Equations by Multistage Adomian Decomposition Method. *Proceedings of the 3rd International Conference on Advances in Applied Science and Environmental Technology (ASET 2015)*. 28-29 December 2015, Bangkok. pp. 79–82.

[8] Ismail, N.I.N., Majid, Z.A. & Senu, N. 2020. Hybrid multistep block method for solving neutral delay differential equations. *Sains Malaysiana*, 49(4), 929–940.

[9] Juhl, A. & Shirokoff, D. 2025. Algebraic conditions for stability in Runge-Kutta methods and their certification via semidefinite programming. *Applied Numerical Mathematics*, 207(1), 136–155.

[10] Kashkari, B.S.H. & Syam, M.I. 2019. Optimization of one step block method with three hybrid points for solving first-order ordinary differential equations. *Results in Physics*, 12, 592–596.

[11] Lambert, J.D. 1974. Computational Methods in Ordinary Differential Equations. *Journal of Applied Mathematics and Mechanics*, 54(7), 523–523.

[12] Mohd Ijam, H., Ibrahim, Z.B. & Mohd Zawawi, I.S. 2024. Stiffly Stable Diagonally Implicit Block Backward Differentiation Formula with Adaptive Step Size Strategy for Stiff Ordinary Differential Equations. *Matematika*, 40(1), 27–47.

[13] Obarhua, F.O. 2023. Three-step Four-point Optimized Hybrid Block Method for Direct Solution of General Third Order Differential Equations. *Asian Research Journal of Mathematics*, 19(6), 25–44.

[14] Olukunle, O.E. & M, O.F. 2019. A Block Scheme with Bernstein Single Step Method for Direct Solution Second Order Differential Equations. *International Journal of Research and Innovation in Applied Science*, IV(XII), 99–103.

[15] Omar, Z. & Abdelrahim, R. 2016. New uniform order single step hybrid block method for

solving second order ordinary differential equations. *International Journal of Applied Engineering Research,* 11(4), 2402–2406.

[16] Omar, Z. & Alkasassbeh, M. F. 2016. Generalized One-Step Third Derivative Implicit Hybrid Block Method for the Direct Solution of Second Order Ordinary Differential Equation. *Applied Mathematical Sciences*, 10(9), 417–430.

[17] Qayyum, M. & Fatima, Q. 2022. Solutions of Stiff Systems of Ordinary Differential Equations Using Residual Power Series Method. *Journal of Mathematics,* 2022, Article ID 7887136.

[18] Qureshi, S., Adebowale, M., Ali, A., Senapon, A., Matthew, O., Mahmoud, W. & Osman, M.S. 2023. A new adaptive nonlinear numerical method for singular and stiff differential problems. *Alexandria Engineering Journal*, 74(1), 585–597.

[19] Ramos, H., Qureshi, S. & Soomro, A. 2021. Adaptive step-size approach for Simpson's-type block methods with time efficiency and order stars. *Computational and Applied Mathematics,* 40(6), 1-20.

[20] Rosli, N., Ariffin, N. A. N., Hoe, Y.S. & Bahar, A. 2019. Stability Analysis of Explicit and Semi-implicit Euler Methods for Solving Stochastic Delay Differential Equations. *In:* Kor, LK., Ahmad, AR., Idrus, Z., Mansor, K. (eds). *Proceedings of the Third International Conference on Computing, Mathematics and Statistics (iCMS2017)*. Singapore: Springer. pp. 179–186.

[21] Rosser, J.B. 1967. Runge-Kutta for all seasons. *Siam Review,* 9(3), 417-452.

[22] Rufai, M.A., Carpentieri, B. & Ramos, H. 2023. A New Hybrid Block Method for Solving First-Order Differential System Models in Applied Sciences and Engineering. *Fractal and Fractional,* 7(10), 1–12.

[23] Shampine, L.F. & Watts, H.A. 1969. Block Implicit One Step Methods. *Mathematics of Computation,* 23, 731–740.

[24] Singh, G., Garg, A., Kanwar, V. & Ramos, H. 2019. An efficient optimized adaptive step-size hybrid block method for integrating differential systems. *Applied Mathematics and Computation,* 362, 124567.

[25] Tassaddiq, A., Qureshi, S., Soomro, A., Hincal, E. & Shaikh, A.A. 2022. A new continuous hybrid block method with one optimal intrastep point through interpolation and collocation. *Fixed Point Theory and Algorithms for Sciences and Engineering,* 2022, Article Number 22.

TRANSACTIONS ON SCIENCE AND TECHNOLOGY