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# Application of Newton-Gauss-Seidel method for solving multi-objective constrained optimization problems

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**ABSTRACT** Many problems in life are composed of conflicting and influencing multiple objectives, and people often encounter the optimization problem of simultaneously optimizing multiple objectives in each area, which is called multi-objective optimization problem. Therefore, solving such problems has important scientific research value and practical significance. This paper proposes a Newton Gauss-Seidel iteration method for solving multi-objective constrained optimization problems by constructing Newton directions and introducing Gauss-Seidel (GS) iterative method for solving linear equations. The solution of this combination between Newton, Gauss-seidel and Lagrange multiplier method involves two stages: objective function and constraint condition processing stage. In the first stage, the original multi-objective function is scalarized, and only the decision-maker needs to give each objective function a weight, by transforming it into a single objective constrained optimization problem. Then the Lagrange multiplier method was used to transform the constrained optimization problem into an unconstrained optimization problem. The second stage is to use the Newton-Gauss-Seidel (NGS) iterative method to solve the transformed constrained optimization problem. Finally, numerical experiments showed that our proposed algorithm can achieve good results.

**KEYWORDS:** Multi-objective constrained optimization; Weighting method; Lagrange multiplier; Newton's method; Gauss-Seidel iteration.
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## INTRODUCTION

Constrained multi-objective optimization problems (CMOP) exist and may be widely encountered in the real world, such as in vehicle path planning (Jozefowiez *et al.*, 2008), traffic route optimization (Zhu & Zhu, 2020), financial investment (El-Abbasy *et al.*, 2020), and so on. In these problems, there are often multiple objectives that need to be optimized, but the objective functions tend to be conflicting and repelling in nature, which often leads to poor results for the other objectives when one objective is optimized. Therefore, it is necessary to design a good algorithm to obtain a set of well-distributed Pareto-optimal solutions. While optimizing, there is also the need to consider multiple constraints, which makes the problem even more complex. For example, in the target space, the feasible region may become discontinuous and difficult to find, or the feasible region may become extremely narrow resulting in difficulty in identifying the optimal solution (Liu, 2017). There are also problems of many local optimization traps that make it difficult for the algorithm to converge to the true Pareto frontier. Therefore, resolving these problems is an arduous and complex task and is currently attracted many attentions due to its wide applications.

## BACKGROUND THEORY

The mathematical model for multi-objective optimization problems with constraints that this paper endeavours to solve can be expressed as

$$min\ F(x) = (f_1(x), f_2(x), \cdots, f_m(x))^T, \tag{1}$$
with
$$s.t.\ g_j(x) \le 0, j = 1, \dots, p,$$
$$h_k(x) = 0, k = p+1, p+2, \dots, p+q,$$
$$x \in R^n,$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$ is an n-dimensional decision vector, $F(x) = (f_1(x), f_2(x), \cdots, f_m(x))^T \in R^m$ is the target vector, $g_j(x) \le 0$ is the $j$-th inequality constraint condition, $h_k(x) = 0$ is the $k$-th equation constraint condition.

It is well known that scalar method has the ability to transform multi-objective optimization problems into single-objective for solution, and is one of the most effective methods for solving multi-objective optimization problems. Typical methods mainly include objective method, linear weighting, minimax, and some other methods (Hu,1990). In addition, there are also hierarchical sorting methods, key target methods, feasible direction methods and others (Lin & Dong, 1992). Among them, the linear weighting method has gradually become a widely used method by scholars (Stanimirovic *et al.*, 2011) in solving multi-objective problems, due to its simplicity and ease of operation. Linear weighting method refers to the method of assigning a coefficient to each objective function of the multi-objective optimization problem according to its importance, which is based on analysing each objective function and then, adding these objective functions with coefficients to construct a single objective function as follows.

$$minF(x) = w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x) \tag{2}$$

where $w_i(i = 1, \dots, m)$ is the weighting coefficients, and $\sum_{i=1}^{m} w_i = 1$.

The indirect method is one of the effective methods for solving constrained optimization problems, which include Lagrange multiplier method (Laptin & Bardadym, 2019), penalty function method (Nguyen & Strodiot,1979; Hassan & Baharum, 2019; Rockafellar, 1973; Napituplu *et al.*, 2018), feasible direction method (Zoutendijk, 1970), gradient projection method (Rosen, 1960), and others. The Lagrange multiplier method, which has a strict mathematical foundation and is more efficient, has been favoured by many researchers. In this paper, the Lagrange multiplier method is mainly used to deal with constraints in multi-objective optimization problems. The specific expression for this method may be represented by Equation (3).

$$L(x, \mu, \lambda) = F(x) + \sum_{j=1}^{p} \mu_j g_j(x) + \sum_{k=p+1}^{p+q} \lambda_k h_k(x) \tag{3}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_p)^T$ and $\lambda = (\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_{p+q})^T$, the latter being the Lagrange multiplier.

Newton's method, as described by Atkinson (1985), is a well-established approach for addressing unconstrained optimization problems. It generates a Newton iterative direction contingent upon the objective function and constraint function being continuously differentiable twice, so that the solution-seeking direction is a descending direction for each objective function under feasible conditions. This method searches for the optimal step size along this direction, and ultimately and iteratively reaches the optimal solution. The iterative formula of the Newton's method can be expressed as

$$x_{k+1} = x_k + \alpha_k d_k, \tag{4}$$

where $d_k = -H_k^{-1} g_k$, is the search direction with step size factor $\alpha_k = 1$, $H_k = \nabla^2 L(x_k)$, $g_k = \nabla L(x_k)$.

Newton's method uses both the information of the first and second-order partial derivatives of the objective function and at the same time, needs the Hessian matrix of the objective function to be positive definite. If the Hessian matrix cannot maintain to be positive definite, the Newton method will become invalid. Therefore, this paper will regard the Newton iteration as a linear system and use the Gauss-Seidel (GS) iteration method for resolving the Newton direction.

## METHODOLOGY

### Newton Iterative Method

In Equation (4), $H_k^{-1}$ is the inverse of the Hessian matrix $H_k = \nabla^2 L(x_k)$, and the iteration direction $d_k = -H_k^{-1} g_k$ can be expressed as

$$H_k d_k = -g_k \quad (5)$$

As long as $H_k$ is positively definite, the Newton direction of Equation (4) will be in the descending direction. Therefore, its inverse exists and satisfies the inequality:

$${g_k}^T d_k = -{g_k}^T H_k^{-1} g_k < 0 \quad (6)$$

As previously outlined, the multi-objective constrained optimization problem uses the weighting method and Lagrange multiplier method to convert it into a single objective unconstrained optimization problem based on Equation (3). In the problem, $L(x, \mu, \lambda)$ is an objective function containing *n+p+q* variables. Because the degree of the Hessian matrix obtained using the Newton method is 1, there must be 0 elements in the diagonal elements of the Hessian matrix. Therefore, the Hessian matrix may be represented as

$$H_k = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,n+p+q} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & a_{2,n+1} & \cdots & a_{2,n+p+q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & a_{n,n+1} & \cdots & a_{n,n+p+q} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & a_{n+1,n+1} & \vdots & a_{n+1,n+p+q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n+p+q,1} & a_{n+p+q,2} & \cdots & a_{n+p+q,n} & a_{n+p+q,n+1} & \cdots & a_{n+p+q,n+p+q} \end{bmatrix} \quad (7)$$

where, the diagonal element $a$ in Equation (7), $a_{1,1}, \cdots, a_{n+p+q,n+p+q}$ contains $p + q$ elements of 0.

### Newton-GS Method

The coefficient matrix $H_k$ can be used to solve the linear system of Equation (5). Therefore, an iterative method can be applied to solve it (Sulaiman *et al.*, 2014; Ghazali *et al.*, 2019), but this method needs to ensure that its diagonal element is not zero, and Equation (5) can be rewritten as

$$H_k' \cdot H_k d_k = -H_k' \cdot g_k$$

which can also be written as

$$Hd = g \quad (8)$$

where, $H = H_k' \cdot H_k$ is an invertible matrix, $g = -H_k' \cdot g_k = [g_1, g_2, g_3, \dots, g_{n+p+q}]^T$ and $d^T = [d_1, d_2, d_3, \dots, d_{n+p+q}]$. $H$ can be further split into

$$H = M - N. \quad (9)$$

$M$ is known as a splitting matrix which is a selectable invertible matrix and makes $Md = g$ easy to solve.

Solving Equation (8) is equivalent to solving $d = M^{-1}Nd + M^{-1}g$, that is, solving a system of linear equations which can be simplified as

$$d = Bd + f \quad (10)$$

As a result, the iterative method of Equation (10) may be applied using the following conditions.

$$\begin{cases} d_0 \ (Initialization\ vector) \\ d_{k+1} = Bd_k + f, k = 0,1, \dots \end{cases} \quad (11)$$

where $B = M^{-1}N$ $and$ $f = M^{-1}g$. $M$ in Equation (9) is a splitting matrix and is the lower triangular part of $H$. Given so, with $M = D - E$ (lower triangular matrix), then $H = M - N = D - E - U$, and the overrelaxation iteration method for solving linear Equations (8) from Equations (10) can be obtained (Sulaiman *et al.*, 2015; Ghazali *et al.*, 2019) as

$$\begin{cases} d_0 \ (Initialization\ vector) \\ d_{k+1} = (D-E)^{-1}Ud_k + (D-E)^{-1}g, k = 0,1, \dots \end{cases} \quad (12)$$

where, $(D - E)^{-1}U$ is the iteration matrix of the GS iteration method for the equation system of Equation (7). Therefore, the formula of the GS iterative method can be used to determine the Newton iterative direction of Equation (4) for the linear system represented by Equation (8), which is a Newton Gauss-Seidal (NGS) iterative algorithm to solve the problem of Equation (1). The iterative approach of the algorithm is listed in Table 1 for reference.

**Table 1.** Algorithm 1 with NGS steps

| | |
|---|---|
| Step 1. | Provide the initial value $x_0$ and the accuracy threshold $\varepsilon_1 = 10^{-5}$， $\varepsilon_2 = 10^{-10}$ and let k: =0. |
| Step 2. | Calculate gradient $g_k$ and matrix $H_k$.<br>If $\|g_k\| < \varepsilon$, that is, the value of the gradient at this point is close to 0, then the extreme point is reached, and the iteration is stopped, otherwise, go to step 3.1. |
| Step 3. | Step 3.1. Calculate the matrix $H = H_k' \cdot H_k$, and determine matrix $D, E, U$.<br>Step 3.2. Calculate the search direction $d_{k+1} = (D-E)^{-1}Ud_k + (D-E)^{-1}g$.<br>Step 3.3. Calculate the convergence condition, that is, if $\|d_{k+1} - d_k\| < \varepsilon_2$, then go to step 4, otherwise go to step 3.2. |
| Step 4. | Calculate the new iteration point as $x_{k+1} = x_k + \alpha_k d_k$. |
| Step 5. | Let $k \coloneqq k + 1$, go to step 2. |

## RESULT AND DISCUSSION

### Symbol Description

For convenience, Table 2 shows the abbreviations of symbols used in the analysis and discussion on the computation results in this paper.

**Table 2.** Description of symbols used in the results.

| Notation | Description |
|---|---|
| M | Method |
| NOI | Number of internal iterations |
| NGS | Newton-GS method |
| TM | computational time (Unit: Second) |
| POS | Pareto optimal solution |
| FOV | Local optimal Value $L(x)$ |

### Numerical Calculation

To test the practical feasibility of the NGS method, a comparative test is undertaken here to test the NGS method. The comparison was done concerning similar results obtained from two references which is by Yuan & Li (2005) and Xiao (2010). The test was performed under equality and inequality constraints, as well as being based on two objective functions and three objective functions. The following two examples will first demonstrate the performance of the NGS method in dealing with equality constraints.

*Example 1* (Yuan & Li, 2005):

$$\min f(x) = (f_1(x), f_2(x), f_3(x))$$
$$s.t. x_1 + 2x_2 - x_3 = 5,$$
$$x_1 - x_2 - x_3 = -1,$$

where $f_1(x) = (x_1 - x_2 + x_3)^2, f_2(x) = {x_1}^2 + 2{x_2}^2 + 3{x_3}^2, f_3(x) = (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2$.

The initial point is $x_0 = (1,1)^T$, and the same weight coefficient $\lambda = (0.5,0.3,02)$. Implementing the NGS iteration algorithm in MATLAB, the Pareto optimal solution obtained is $x^* = (1.3611, 2.0, 0.3611)^T$ (Miettinen, 2004), and the optimal objective function is $f(x^*) = (0.07716,10.2438,7.0942)^T$ and $L(x^*) = 4.5306$ . Table 3 shows the numerical results of the calculation.

**Table 3.** Calculation results of Example 1.

| M | NOI (outer) | NOI (Inner) | TM | POS | FOV $\boldsymbol{L}(\boldsymbol{x}^*)$ |
|---|---|---|---|---|---|
| NGS | 1 | 53 | 2.505895 | (1.3611, 2.0, 0.3611) | 4.5306 |

*Example 2* (Yuan & Li, 2005):

$$\min f(x) = \left(f_1(x), f_2(x)\right),$$
$$s.t. 2x_1 + 5x_2 = 25,$$

where $f_1(x) = 1 - (x_1)^2 + e^{-x_1 - x_2} + {x_2}^2 - 2x_1x_2, f_2(x) = e^{x_1} - 3x_2$.

The initial point used is $x_0 = (1,1)^T$, and the weight coefficient $\lambda = (0.5,0.5)$. Similarly, using the NGS iteration algorithm in MATLAB , the Pareto optimal solution calculated is $x^* = (2.5653, 3.9739)^T$, where the optimal objective function is $f(x^*) = (-10.1787,1.0827)^T$ and $L(x^*) = -4.54798$. The numerical results of the calculation are shown in Table 4.

**Table 4.** Calculation results of Example 2.

| M | NOI (Outer) | NOI (Inner) | TM | POS | FOV $\boldsymbol{L}(\boldsymbol{x}^*)$ |
|---|---|---|---|---|---|
| NGS | 7 | 146 | 9.048878 | (2.5653, 3.9739) | −4.54798 |

To demonstrate the feasibility of the NGS method in dealing with multi-objective optimization problems with inequality constraints, the following examples demonstrate are given.

*Example 3* (Xiao, 2010)

$$\min f(x) = (f_1(x), f_2(x))$$
$$s.t. {x_1}^2 + {x_2}^2 \leq 4,$$
$$(x_1 - 2)^2 + {x_2}^2 \leq 4,$$

where $f_1(x) = (x_1 + x_2)^2$and $f_2(x) = x_2$.

The initial point $x_0 = (-1, -1)^T$, and elect the weight coefficient $\lambda = (0.5,0.5)$. The NGS iteration algorithm and MATLAB software, the Pareto optimal solution is calculated as $x^* = (1.0, -1.7321)^T$, where the optimal objective function is $f(x^*) = (0.5359, -1.7321)^T$ and $L(x^*) = -0.59808$. The numerical results of the calculation are shown in Table 5.

**Table 5.** Calculation results of Example 3.

| M | NOI (Outer) | NOI(Inner) | TM | POS | FOV $\boldsymbol{L}(\boldsymbol{x}^*)$ |
|---|---|---|---|---|---|
| NGS | 6 | 4181 | 80.3247 50 | (1.0, -1.7321) | $-0.5981$ |

*Example 4* (Xiao, 2010)

$$\begin{gathered} min f(x) = (f_1(x), f_2(x)) \\ s.t. 2x_1 + x_2 \geq 1, \\ x_1 + 3x_2 \geq 1, \\ x_1, x_2 \geq 0. \end{gathered}$$

where $f_1(x) = x_1 + x_2$ and $f_2(x) = x_1{}^2 + 9x_2{}^2$.

The initial point is $x_0 = (-1, -1)^T$, and the weight coefficient $\lambda = (0.5, 0.5)$. The NGS iteration algorithm yields the Pareto optimal solution of $x^* = (0.4000, 0.2000)^T$, where the optimal objective function obtained is $f(x^*) = (0.6000, 0.5200)^T$ and $L(x^*) = 0.56$. The numerical results of the calculation are shown in Table 6.

**Table 6.** Calculation results of Example 4.

| **M** | **NOI (Outer)** | **NOI (Inner)** | **TM** | **POS** | **FOV $\boldsymbol{L}(\boldsymbol{x}^*)$** |
|---|---|---|---|---|---|
| NGS | 1 | 541 | 16.434790 | (0.4000,0.2000) | 0.56 |

**Results Comparison**

The numerical examples in the above shows that the devised algorithm can efficiently and effectively solve multi-objective constrained optimization problems. From Examples 1 and 2 with equality constraints, using the NGS method compared to the method of Yuan & Li (2005), it does not require the use of dimensionality reduction steps in obtaining a Pareto effective solution. Meanwhile, from the results of Examples 3 and 4 with inequality constraints, the NGS method can obtain an effective solution without the need for interactive iteration steps as described by Xiao (2010). This demonstrates that the devised method exhibits certain efficiency and advantages in calculating and solving multi-objective optimization problems.

## CONCLUSION

This article studies the algorithms for multi-objective constrained optimization problems. It uses linear weighting and Lagrange multiplier methods to transform the constrained multi-objective optimization problem into an unconstrained single-objective optimization problem. Based on the transformed problem, an NGS iteration method is used as a computational technique to obtain a Pareto effective solution. This article innovatively combines the linear weighting method, Lagrange multiplier method, and Newton iteration method for solving unconstrained optimization problems. The numerical experiments reported here have demonstrated the effectiveness and superiority of the algorithm. How to apply the algorithm to handle large-scale multi-objective optimization problems will be the direction of further research in the future.

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