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Variants of differential transform method in solving Schrodinger equations

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ABSTRACT This paper obtains analytical solutions for the Schrodinger equations (SEs) using variants of the differential transform method (DTM). The solutions produced by two-dimensional DTM (2D-DTM), reduced DTM (RDTM), and multistep RDTM (MsRDTM) were observed. The outcomes show that the MsRDTM generated more highly accurate solutions to SEs than the 2D-DTM and RDTM. The solutions also show that the MsRDTM is straightforward to use, saves a significant amount of computing work when solving SEs, and has potential for broad application in other complex partial differential equations. Graphical representations are presented to illustrate the different effectiveness and accuracy of the variants of DTM.

KEYWORDS: Differential transform method; Two-dimensional differential transform method; Reduced differential transform method; Multistep reduced differential Transform Method; Schrodinger Equation.
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Original Article

INTRODUCTION

Schrodinger equations (SEs) appear in a variety of fields, including fluid mechanics, quantum mechanics, nonlinear optics, biology, and other disciplines (Wazwaz, 2006). This equation has also been used to model other scientific phenomena, such as optics, and light emission in cables of fibre optics. This equation has been solved numerically and analytically using various methods, including the Bernoulli (G′/G)-expansion methods (Gu & Aminakbari, 2022), and the modified exponential Jacobi method (Nonlaopon *et al.*, 2022).

One of the most used approximate analytical methods in solving this type of equation is the differential transform method (DTM). With regards to the developments of the DTM and its variants, the two-dimensional DTM (2D-DTM) was initially introduced by Zhou (1986) in solving linear and nonlinear initial valued problems in electrical circuit analysis. Then, Ravi Kanth and Aruna (2009) implemented the 2D-DTM in solving SEs. The reduced DTM (RDTM) was then introduced by Keskin *et al.* (2011). It caught massive attention from researchers since it helped solving a variety of problems. Analytical approximations, often exact solutions, are provided by this approach in a rapidly converging power series form with computed elements (Deresse, 2022). Not only does it reduce its calculations, but it also solves the problems without the need of complicated steps.

Odibat *et al.* (2010) introduced the multistep DTM (MsDTM) and applied it to various systems. The method produces solution where its convergent series rapidly converges in a large time frame which then improves the convergence of the series solution. This multistep scheme was then utilized by Al-Smadi *et al.* (2017) by applying it to RDTM, resulting in the multistep RDTM (MsRDTM). Hussin *et al.* (2018; 2019) proposed and implemented the Multistep Modified RDTM (MMRDTM) to obtain solutions of NLSEs and fractional NLSEs (FNLSEs) respectively.

This study aims to show the differences of DTM and its variants in solving SEs. The differences are observed through the different solutions of DTM, RDTM, and MsRDTM opposing the exact solutions provided. The finding indicates great accuracy of the MsRDTM in solving SEs than the previous methods, 2D-DTM, and RDTM. The remaining sections of this work are organized as follows. The formulation of 2D-DTM in Section 2, the formulation of RDTM in section 3, while the formulation of MsRDTM in section 4. Section 5 illustrates the application of the methods in an example of SEs with analytical answers, which are presented in tables and graphs. Lastly, section 6 provides conclusion.

METHODOLOGY

Two-Dimensional Differential Transform Method

Consider two-variable function $u(x,t)$, be analytical in the domain K, and let $(x,t) = (x_0,t_0)$ in this domain. Function $u(x,t)$ is then represented by one series whose centre located (x_0,t_0) as (Ravi Kanth & Aruna, 2009)

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{x=x_0,t=t_0}(x-x_0)^k(t-t_0)^h \tag{1}$$

The differential transform of the function $u(x,t)$ is in the form

$$U_{k,h}(x,t) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{x=x_0,t=t_0} \tag{2}$$

where $u(x,t)$ is the original function and $U_{k,h}(x,t)$ is the transformed function. Given the following is the differential inverse transform of $U_{k,h}(x,t)$

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{k,h}(x,t)(x-x_0)^k(t-t_0)^h. \tag{3}$$

When (x_0,t_0) are taken as $(0,0)$, Equation (3) can be expressed as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{k,h}(x,t)x^k t^h. \tag{4}$$

Reduced Differential Transform Method

Consider the two-variable function $u(x,t)$ may be written as the product of two single-variable functions: $v(x,t) = f(x)g(t)$. On the foundational properties of the one-dimensional differential transform, the function $u(x,t)$ may be written as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{j=0}^{\infty}G(j)t^j\right) = \sum_{k=0}^{\infty}U_k(x)t^k \tag{5}$$

where the t-dimensional span function of $u(x,t)$ is denoted by $U_k(x)$. The fundamental definitions of RDTM, as stated by Keskin *et al.* (2011), are as follows: Definition 1 stated that if the domain of interest's function $u(x,t)$ is analytical and continuously differentiable with regard to time t and space x, then letting

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} \tag{6}$$

where the transformed function is the t-dimension span function $U_k(x)$. In this study, the primary function is denoted by the small letter $u(x,t)$, while the altered function is symbolized by the capital letter $U_k(x)$.

Definition 2 states the following differential inverse transform of $U_k(x)$:

$$u(x,t) = \sum_{k=0}^{\infty}U_k(x)t^k. \tag{7}$$

Then, by fusing (6) and (7), we obtain

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}t^k. \tag{8}$$

According to the preceding definitions, the RDTM concept is obtained from the expanded power series. Consider the following operator-form nonlinear partial differential equation to explain the fundamental RDTM concepts

$$\mathcal{L}u(x,t) + \mathcal{R}u(x,t) + \mathcal{N}u(x,t) = g(x,t) \tag{9}$$

with initial condition

$$u(x,0) = f(x), \tag{10}$$

where $\mathcal{L} = \frac{\partial}{\partial t'}$, $\mathcal{R}$ is a partial derivatives linear operator, $\mathcal{N}u(x,t)$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term.

The RDTM suggests that the iteration formula shown below can be formulated as:

$$(k + 1)U_{k+1}(x) = \mathcal{G}_k(x) - \mathcal{R}U_k(x) - \mathcal{N}U_k(x) \tag{11}$$

where $U_k(x), \mathcal{R}U_k(x), \mathcal{N}U_k(x)$ *and* $\mathcal{G}_k(x)$ are the transformations of the functions $\mathcal{L}u(x,t), \mathcal{R}u(x,t), \mathcal{N}u(x,t)$ *and* $g(x,t)$ respectively.

Based on initial condition (10), we write

$$U_0(x) = f(x). \tag{12}$$

The following $U_k(x)$ values are obtained by substituting Equation (8) into (7) and doing a simple iterative computation. The n-terms approximation solution is then obtained as follows by applying inverse transformation on the set of values $\{U_k(x)\}_{k=0}^{n}$:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{13}$$

As a result, the problem's exact solution is provided by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t). \tag{14}$$

Multistep Reduced Differential Transform Method

While applying the concepts of RDTM from Equation (5) (13), the multistep scheme is as follows. Divide the interval $[0,T]$ to generate R subintervals $[t_{r-1}, t_r]$ by equal step size $s = \frac{T}{R}$ and nodes $t_r = rs$ such that for $r = 1,2, \dots, R$. The upcoming procedures are used to compute MsRDTM. Firstly, RDTM is applied to the initial value problem of interval $[0, t_1]$. Then by using the initial conditions

$$u(x,0) = f_0(x), \tag{15}$$

we obtain the approximate result

$$u_1(x,t) = \sum_{k=0}^{\mathrm{k}} U_{k,1}(x)t^k, t \in [0, t_1]. \tag{16}$$

At each subinterval $[t_{r-1}, t_r]$, the initial conditions

$$u_r(x, t_{r-1}) = u_{r-1}(x, t_{r-1}) \tag{17}$$

are used for $r \geq 2$ and the implementation of RDTM to the initial value problem on $[t_{r-1}, t_r]$, where t_{r-1} replaces t_0. For $r = 1,2, \dots, R$, the repetition of the process is performed and carried out to construct an approximate solutions sequence $u_r(x,t)$ such as,

$$u_r(x,t) = \sum_{k=0}^{K} U_{k,r}(x)(t - t_{r-1})^k, t \in [t_{r-1}, t_r]. \tag{18}$$

Finally, the MsRDTM proposes the following solutions:

$$u(x,t) = \begin{cases} u_1(x,t), & for\ t \in [0, t_1] \\ u_2(x,t), & for\ t \in, [t_1, t_2] \\ \vdots & \\ u_R(x,t), & for\ t \in, [t_{R-1}, t_R] \end{cases}. \tag{19}$$

It is crucial to note that when the step size $s = T$, the RDTM is derived from MsRDTM.

NUMERICAL RESULT AND DISCUSSION

Consider the numerical example given to observe the differences of DTM, RDTM, and MsRDTM in solving SEs. Linear Schrodinger Equation (LSE) of the form (Ravi Kanth & Aruna, 2009)

$$u_t + \mathrm{i}u_{xx} = 0 \tag{20}$$

is considered with initial condition $u(x,0) = 1 + \cosh 2x$. $1 + \cosh(2x)\, e^{-4it}$ is the exact solution.

By applying the DTM, RDTM, and MsRDTM to equation (20) and using their respected fundamental properties, we have

$$U_{k,h+1}(x) = \left(\frac{-I}{h+1}\right)\Big((k+1) \quad (k+2) \quad \big(U_{k+2,h}(x)\big)\Big), for\ t \in [0,T] \tag{21}$$

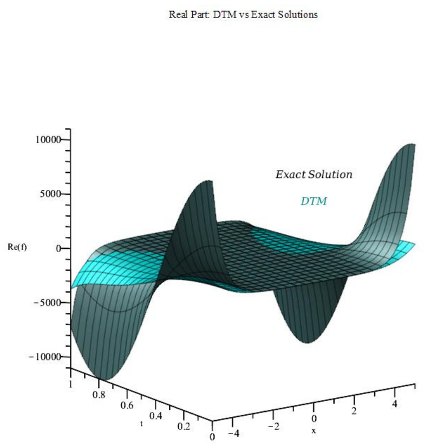
$$U_{k+1}(x) = \left(\frac{-I}{k+1}\right)\left(\frac{\partial^2}{\partial x^2}U_k(x)\right), for\ t \in [0, T] \quad (22)$$

$$U_{k+1,\mathrm{r}}(x) = \left(\frac{-I}{k+1}\right)\left(\frac{\partial^2}{\partial x^2}U_{k,r}(x)\right), for\ t \in [0, t_1]. \quad (23)$$

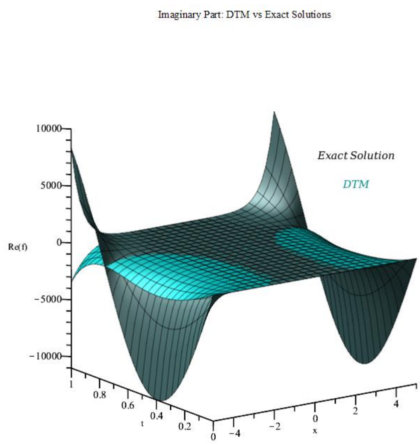
The results of the exact solution, approximate solutions DTM, RDTM, and MsRDTM for $t \in [0,1]$ $and\ x \in [-5,5]$, which involves the real part and imaginary part, are shown in Figures 1(a) - (f), respectively. The MsRDTM solutions for this LSE are therefore proved to be quite near to the exact solutions. In Table 1, the performance error analysis produced by DTM, and its variants are presented. Based on Table 1, the numerical results for absolute error and error norms, L_2 and L_∞ from MsRDTM are significantly smaller which proves its accuracy than DTM and RDTM.

Table 1. Error Analysis of Semi-Analytic Solution for DTM, RDTM, and MsRDTM.

T	Exact Solutions	Absolute Error DTM	Absolute Error RDTM	Absolute Error MsRDTM
0	4.762195691	6.640135×10^{-3}	0	0
0.1	4.465211703 - 1.465068009I	$6.639717045 \times 10^{-3}$	$1.221564085 \times 10^{-6}$	$1.221564085 \times 10^{-6}$
0.2	3.621146980 - 2.698833994I	$6.540919381 \times 10^{-3}$	$1.559385047 \times 10^{-4}$	$2.443817710 \times 10^{-6}$
0.3	2.363260783 - 3.506513433I	$4.538455570 \times 10^{-3}$	$2.651492475 \times 10^{-3}$	$3.666185484 \times 10^{-6}$
0.4	0.8901456830 - 3.760591502I	$1.319803385 \times 10^{-2}$	$1.973012085 \times 10^{-2}$	$4.886638138 \times 10^{-6}$
0.5	-0.565625835 - 3.420954861I	$8.658451228 \times 10^{-2}$	$9.327173470 \times 10^{-2}$	$6.108126554 \times 10^{-6}$
0.6	-1.774219459 - 2.541224667I	$3.244401985 \times 10^{-1}$	$3.307241111 \times 10^{-1}$	$7.329369482 \times 10^{-6}$
0.7	-2.544824830 - 1.260290975I	$9.551882539 \times 10^{-1}$	$9.610715118 \times 10^{-1}$	$8.551244237 \times 10^{-6}$
0.8	-2.755780304 + 0.2196149509I	2.406816598	2.413230176	$9.772797407 \times 10^{-6}$
0.9	-2.373780650 + 1.664848505I	5.408457642	5.417867014	$1.099458958 \times 10^{-5}$
1.0	-1.459135214 + 2.847239087I	11.11508584	11.13228416	$1.221626170 \times 10^{-5}$
	L_2	12.63387264	12.65491392	$2.397003666 \times 10^{-5}$
	L_∞	11.11508584	11.13228416	$1.221626170 \times 10^{-5}$

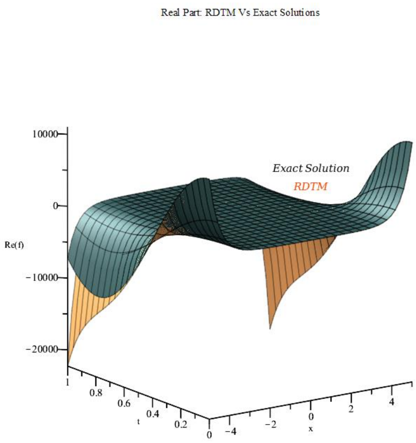


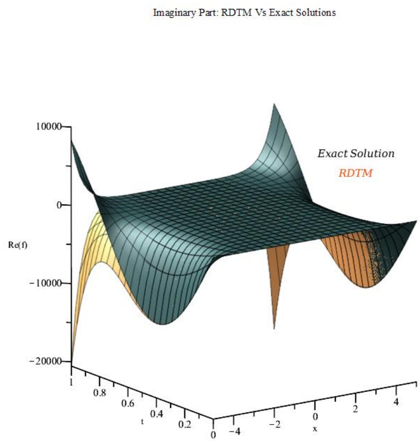
(a)



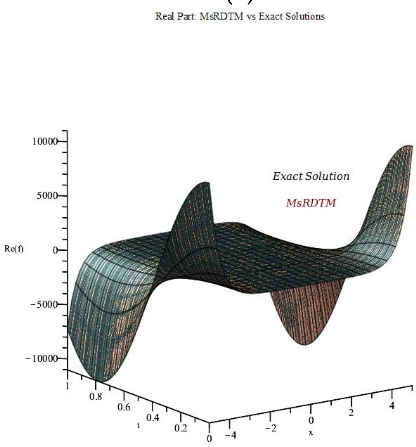
(b)

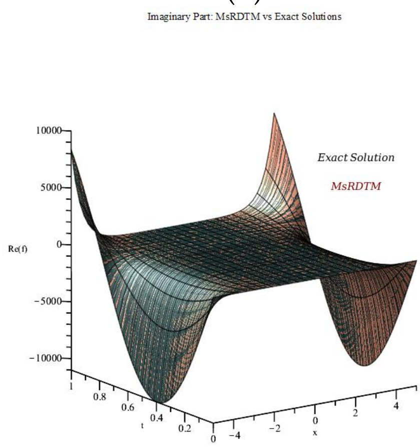
Figure 1. Figure 1(a) and Figure 1(b) are between DTM and exact solution (ES), Figure 1(c) and Figure 1(d) are between RDTM and ES, while Figure 1(e) and Figure 1(f) are between MsRDTM and ES which involve the real part and imaginary part respectively.





(c) (d)





(e) (f)

Figure 1. Figure 1(a) and Figure 1(b) are between DTM and exact solution (ES), Figure 1(c) and Figure 1(d) are between RDTM and ES, while Figure 1(e) and Figure 1(f) are between MsRDTM and ES which involve the real part and imaginary part respectively (*continue*).

CONCLUSION

This work uses the classical DTM, RDTM, and MsRDTM for dealing with SEs. The findings demonstrate the effectiveness and dependability of the MsRDTM, as shown by the outcomes and the graphical representations. Thus, MsRDTM is a valuable mathematical approach for dealing with SEs since it is more accurate than the 2D-DTM, and the RDTM. This paper's calculations were all performed using Maple 2021.

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